Finite difference-based lattice Boltzmann simulation of natural convection heat transfer in a horizontal concentric annulus

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Abstract

In this paper, a finite difference-based lattice BGK model for thermal flows is proposed based on the double-distribution function approach. We applied this model to simulate natural convection heat transfer in a horizontal concentric annulus bounded by two stationary cylinders with different temperatures. Velocity and temperature distributions as well as Nusselt numbers were obtained for the Rayleigh numbers ranging from $2.38 \times 10^3$ to $1.02 \times 10^5$ with the Prandtl number around 0.718. It is found that the numerical results are in good agreement with the experimental and numerical results reported in the literature.

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1. Introduction

Natural convection in annular cavities bounded by co-axial, horizontal cylinders has been investigated widely in the past owing to a number of practical applications associated with this geometry such as heat transfer and fluid flow in nuclear reactors, thermal storage systems,
electrical transmission cables, and electronic component cooling among others. Early studies of this problem can be found in Kuehn and Goldstein [1]. More recently, the effects of eccentricity of the inner cylinder [1–3], and temperature-dependent properties have been investigated [3]. Among the previous studies pertinent to this problem, in addition to experimental investigations, numerical studies, including the finite-difference method, the finite-volume method, and the finite element method, have been predominated [2–4].

The lattice Bhatnagar–Gross–Krook (LBGK) method, a novel kinetic-based numerical approach for simulating fluid flows and associated transport phenomena, has developed rapidly since its emergence. Many studies have made great strides in constructing its theoretical foundation [5,6,17,32] and improving its numerical performance [7–13] over the last decade. Unlike conventional numerical schemes, which discretize the macroscopic governing equations directly, the LBGK method solves the kinetic equation at the mesoscopic scale, i.e. the Boltzmann equation with the BGK assumption [5,6]. Historically, the LBGK method originated from the lattice-gas cellular automata method (LGCA) [14,15], a microscopic model for fluid systems where the imaged fluid particles collide and move on a regular lattice, and indeed it is very similar to the LGA method, except that particles residing on the lattice are replaced by the corresponding distribution functions and the collision operator is approximated by the BGK assumption. But later it was realized that LBGK could also be viewed as a special finite difference scheme of the continuous Boltzmann equation on a regular lattice [5,6], which also defines the associated discrete particle velocities. From this viewpoint, discretization for the particle velocity can be decoupled from the spatial discretization, since the particle velocity in the Boltzmann equation is independent of the particle position [7]. This implies that we can discretize the continuous velocity space into a set of discrete velocities with sufficient symmetry (physical symmetry), while the usual spatial space may be discretized in some curvilinear coordinates. Motivated by this rationale, some people [9–13] have proposed alternative LBGK methods based on finite difference, finite volume, and finite element techniques.

Our literature review shows that most of previous finite difference-based LBGK models are for isothermal flows. In this paper, we aim to extend the finite difference-based LBGK (FDLBGK) model to convection heat transfer. In general, there exist two approaches for incorporating the heat transfer effect into the LBGK method in the literature: the multi-speed approach and the double-distribution function approach. In the first approach, the evolution equation is just the same as that for isothermal flows, except with an enlarged set of discrete velocities and an equilibrium distribution function (EDF) involved higher-order velocity terms. In the second approach, an independent LBGK equation for a newly defined distribution function for energy (or temperature) is introduced, while the LBGK model for the velocity field is basically the same as the one for isothermal flows. The coupling between the two LBGK equations can be implemented in different ways [16–25]. The double-distribution function approach is usually adopted in practical applications, because the multi-speed approach suffers from severely numerical instability and limits in a rather narrow temperature range. In the present paper, following the double-distribution function idea, we extend the finite difference LBGK (FDLBGK) method proposed in a recent work by Guo and Zhao [10] to convection heat transfer and then present the numerical results for the natural convection heat transfer in a horizontal concentric annulus. The numerical results are compared with previous experimental and numerical data, and the advantages of this model over others are also discussed.
The rest of the article is organized as follows: In Section 2, a discrete velocity Boltzmann model for convection heat transfer is presented based on the idea of double-distribution function [16–25]. In Section 3, we propose a finite-difference-based LBGK model for heat transfer based on the discrete velocity model in curvilinear coordinates following the procedure developed in [10]. In Section 4, numerical results for the natural convection in a horizontal concentric annulus using the finite difference-based LBGK model are presented and discussed. And finally some conclusions are drawn in Section 5.

2. The Boltzmann BGK equation

In kinetic theory, the evolution of the single-particle density distribution function \( f(t, \vec{x}, \vec{c}) \), which represents the probability density of a particle with unit mass moving with velocity \( \vec{c} \) at point \( \vec{x} \) at time \( t \), is governed by the Boltzmann equation [26],

\[
\frac{\partial f}{\partial t} + \vec{c} \cdot \nabla f = \frac{\partial}{\partial t} e f,
\]

where the term on the right hand side represents the collision operator. An approximation of this complicated operator is the so-called BGK or the single-relaxation time model [27]. With the BGK approximation, the Boltzmann equation can be rewritten as

\[
\frac{\partial f}{\partial t} + \vec{c} \cdot \nabla f = -\frac{1}{\tau} (f - f^{\text{eq}}),
\]

where \( \tau \) is the relaxation time, and \( f^{\text{eq}} \) denotes the local equilibrium distribution function (EDF) satisfying the Maxwell–Boltzmann distribution [26]:

\[
f^{\text{eq}} = \rho (2\pi RT)^{-D/2} \exp \left[ -\frac{(\vec{c} - \vec{u})^2}{2RT} \right],
\]

where \( D \) denotes the space dimension, \( R \) is the gas constant; \( \rho \), \( \vec{u} \), and \( T \) are the macroscopic density, velocity, and temperature, respectively, which are defined by the velocity moments of the density distribution function:

\[
\rho = \int f \, d\vec{c},
\]

\[
\rho \vec{u} = \int f \vec{c} \, d\vec{c},
\]

\[
\frac{D\rho RT}{2} = \int \frac{1}{2} f (\vec{c} - \vec{u})^2 \, d\vec{c}.
\]

Compared with the original Boltzmann equation, the BGK equation (2) not only retains the most basic features of the original Boltzmann collision operator, but also has a much simpler form. Therefore, it has been regarded as an appropriate kinetic base for many numerical algorithms, including the LBM and the gas-kinetic BGK scheme [28–31].
It is also noted that when the macroscopic velocity $\vec{u}$ is small, the EDF given by Eq. (3) can be expanded as a Taylor series up to $\vec{u}^2$:

$$f^{\text{eq}} = \rho \left( \frac{1}{2\pi RT} \right)^{D/2} \exp \left( -\frac{\vec{c}^2}{2RT} \right) \left[ 1 + \frac{(\vec{c} \cdot \vec{u})}{RT} + \frac{(\vec{c} \cdot \vec{u})^2}{2R^2T^2} - \frac{\vec{u}^2}{2RT} \right].$$

(7)

We now introduce a temperature distribution function $q$:

$$q = \frac{\langle \vec{c} - \vec{u} \rangle^2}{DR} f.$$

(8)

Note that the denominator of Eq. (8) is different from the internal energy density distribution function $g$ defined by He et al. [17]. Temperature can thus be obtained from:

$$\rho T = \int q \, d\vec{c}$$

(9)

and the evolution equation for $q$ can be derived from the Boltzmann equation (1) as follows:

$$\frac{\partial q}{\partial t} + \vec{c} \cdot \nabla q = -\frac{1}{\tau_s} (q - q^{\text{eq}}) - f \frac{2(\vec{c} - \vec{u})}{DR} \cdot \left[ -\nabla p + \nabla \cdot P \right] - f \frac{2}{DR} (\vec{c} - \vec{u}) (\vec{c} - \vec{u}) : \nabla \vec{u},$$

(10)

where a new relaxation time $\tau_s$ is introduced for simplifying the collision term, and the EDF is defined as $q^{\text{eq}} = \langle \vec{c} - \vec{u} \rangle^2 f^{\text{eq}}/DR.$ We have recently shown that in the limit of incompressibility ($Ma \ll 1$) and for the cases when the compression work and viscous heat dissipation are negligibly small, Eq. (10) can be simplified to [32]

$$\frac{\partial q}{\partial t} + \vec{c} \cdot \nabla q = -\frac{1}{\tau_s} (q - q^{\text{eq}}).$$

(11)

We have also shown that the macroscopic conservation equations of mass, momentum, and energy can be recovered from the Boltzmann BGK equations (2) and (11) through the Chapman-Enskog expansion as [32]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0,$$

(12)

$$\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u} \vec{u}) = -\nabla p + \nabla \cdot (\rho v (\nabla \vec{u} + \vec{u} \nabla)),$$

(13)

$$\frac{\partial (\rho T)}{\partial t} + \nabla \cdot (\rho \vec{u} T) = \nabla \cdot (\rho \vec{c} T),$$

(14)

where $p = \rho RT$ is the pressure, and the shear viscosity $v$ and thermal diffusivity $\alpha$ are given by

$$v = \tau RT,$$

(15)

$$\alpha = \frac{D + 2}{D} \tau_sRT.$$

(16)

Note that the velocity spaces of the Boltzmann BGK equations (2) and (11) are continuous. The corresponding discrete velocity equation, Eq. (2), can be obtained following the procedure proposed by He and Luo [5,6]. For the two-dimensional case, the resultant discrete velocity equations are given by
\[ \frac{\partial f_i}{\partial t} + \bar{c}_i \cdot \nabla f_i = -\frac{1}{\tau} (f_i - f_i^{eq}), \]  
where \[ f_i = f(t, \bar{x}, \bar{c}_i) \] and the discrete velocities \( c_i \) are given by [33]

\[ \bar{c}_i = \begin{cases} (0, 0), & i = 0, \\ c(\cos[(i - 1)\pi/2], \sin[(i - 1)\pi/2]), & i = 1, 2, 3, 4, \\ \sqrt{2}c(\cos[(2i - 9)\pi/4], \sin[(2i - 9)\pi/4]), & i = 5, 6, 7, 8, \end{cases} \]  
with \( c = \sqrt{3RT} \). The corresponding discrete EDF, \( f_i^{eq} \), can be obtained from Eq. (3) by performing a Taylor expansion up to \( \bar{u}^2 \) as follows:

\[ f_i^{eq} = w_i \rho \left[ 1 + \frac{(\bar{c}_i \cdot \bar{u})}{c_s^2} + \frac{(\bar{c}_i \cdot \bar{u})^2}{2c_s^4} - \frac{\bar{u}^2}{2c_s^2} \right], \]  
where \( c_s = \sqrt{RT} \) is the sound speed, and \( w_i = 4/9, w_i = 1/9 \) for \( i = 1–4 \), and \( w_i = 1/36 \) for \( i = 5–8 \). The macroscopic variables are now defined by

\[ \rho = \sum_{i=0}^{8} f_i, \]  
\[ \rho \bar{u} = \sum_{i=0}^{8} f_i \bar{c}_i. \]

In a similar manner, performing a Taylor expansion of \( q^{eq} \) up to \( \bar{u}^2 \), we can obtain the EDF for temperature:

\[ q^{eq} = \rho T \left( \frac{1}{2\pi RT} \right)^{D/2} \exp \left( -\frac{\bar{c}^2}{2RT} \right) \left[ \frac{\bar{c}^2}{DRT} + \left( \frac{\bar{c}^2}{DRT} - \frac{2}{D} \right) \frac{(\bar{c} \cdot \bar{u})}{RT} \right. \right. 
\[ \left. + \left( \frac{\bar{c}^2}{DRT} - \frac{4}{D} \right) \frac{(\bar{c} \cdot \bar{u})^2}{2R^2T^2} - \left( \frac{\bar{c}^2}{DRT} - \frac{2}{D} \right) \frac{\bar{u}^2}{2RT} \right], \]  
which can be simplified to

\[ q^{eq} = \rho T \left( \frac{1}{2\pi RT} \right)^{D/2} \exp \left( -\frac{\bar{c}^2}{2RT} \right) \left[ 1 + \frac{(\bar{c} \cdot \bar{u})}{RT} + \frac{(\bar{c} \cdot \bar{u})^2}{2R^2T^2} - \frac{\bar{u}^2}{2RT} \right]. \]  
It has been proved [32] that the above simplification does not alter the corresponding macroscopic conservation equation of energy, Eq. (14). The only change is the value of the thermal diffusivity \( \alpha \) from \((D + 2)\nu/\tau DRT\) to \( \tau DRT \). The difference can be corrected by adjusting the parameter \( \tau \) in the numerical simulation.

Apparently, a comparison of Eq. (7) with (23) leads to

\[ q^{eq} = T \bar{f}^{eq}. \]  
It follows from Eq. (24) that the discrete velocity set, given by Eq. (18), can also be used to discretize the velocity space of the Boltzmann BGK equation (11) to yield
where \( q_i = q(t, \bar{x}, \bar{u}) \) and the corresponding EDF is given by
\[
q_{i}^{\text{eq}} = T f_{i}^{\text{eq}}.
\]

The temperature can be obtained from
\[
\rho T = \sum_{i=0}^{8} q_i.
\]

It is worth mentioning that the EDF for temperature given by Eq. (26) was also used in many previous thermal models (e.g. [18]).

For small-Mach number flows, the above discrete velocity Boltzmann BGK model can be simplified by introducing the distribution function for temperature as
\[
\bar{q}_i = \frac{q_i}{\rho}.
\]

And the discrete velocity equations of \( \bar{q}_i \) can be obtained from Eq. (25) as
\[
\frac{\partial \bar{q}_i}{\partial t} + \bar{c}_i \cdot \nabla \bar{q}_i = -\frac{1}{\tau_i} (q_i - q_{i}^{\text{eq}}),
\]
where
\[
q_{i}^{\text{eq}} = w_i T \left[ 1 + \frac{(\bar{c}_i \cdot \bar{u})}{c_s^2} + \frac{(\bar{c}_i \cdot \bar{u})^2}{2c_s^4} - \frac{\bar{u}^2}{2c_s^2} \right].
\]

In fact, the EDF given Eq. (30) can be further simplified by neglecting the terms of \( O(\bar{u}^2) \) to give [32]
\[
q_{i}^{\text{eq}} = w_i T \left[ 1 + \frac{(\bar{c}_i \cdot \bar{u})}{c_s^2} \right].
\]

With this new temperature EDF, temperature can be obtained from
\[
T = \sum_{i=0}^{8} \bar{q}_i.
\]

As a result, the discrete velocity Boltzmann BGK equations (17) and (29) recover the continuity equation (12), momentum conservation equation (13) and the following temperature equation:
\[
\frac{\partial T}{\partial t} + \bar{u} \cdot \nabla T = \nabla \cdot (\alpha \nabla T),
\]
where the thermal diffusivity is \( \alpha = \tau_s R T \).

3. Finite difference-based lattice BGK formulation

Base on the discrete velocity Boltzmann BGK equations (17) and (29) presented in the preceding section, two finite-difference-based LBGK equations for the velocity and temperature fields are derived in this section. This is accomplished by integrating (17) and (29) with respect to time.
and discretizing the spatial derivatives with some standard finite-difference schemes in curvilinear coordinates. For simplicity, hereafter we use the symbols $q_i$ and $q_i^{eq}$ to replace $\tilde{q}_i$ and $\tilde{q}_i^{eq}$ in the rest of the paper.

In the presence of a body force $\tilde{F}$, the discrete BGK equation (17) can be rewritten as
\[
\frac{\partial f_i}{\partial t} + \tilde{c}_i \cdot \nabla f_i = -\frac{1}{\tau} (f_i - f_i^{eq}) + F_i,
\]
where
\[
F_i = \frac{w_i \tilde{c}_i \cdot \tilde{F}}{c_s^2}
\]
is the forcing term [19]. The discrete BGK equation (34) can be solved using any standard numerical schemes in general curvilinear coordinates. In what follows, we take the polar coordinates as an example. Extensions to other curvilinear coordinates are straightforward. In polar coordinates $(r, \varphi)$, Eq. (34) can be rewritten as
\[
\frac{\partial f_i}{\partial t} + c_r \frac{\partial f_i}{\partial r} + \frac{c_\varphi}{r} \frac{\partial f_i}{\partial \varphi} = -\frac{1}{\tau} (f_i - f_i^{eq}) + \frac{w_i \tilde{c}_i \cdot \tilde{F}}{c_s^2}.
\]

Integrating Eq. (36) with respect to time and using the scheme proposed by Guo and Zhao [10] leads to the following semi-discretized equation:
\[
f_i^{n+1} - f_i^n + \Delta t \left[ \frac{1}{r} c_\varphi \frac{\partial f_i^n}{\partial \varphi} + c_r \frac{\partial f_i^n}{\partial r} \right] = \Delta t \left[ -\frac{(1-\theta)}{\tau} (f_i - f_i^{eq}) - \frac{\theta}{\tau} (f_i - f_i^{eq})^{n+1} \right] + \Delta t w_i \tilde{c}_i \cdot \tilde{F},
\]
where $0 \leq \theta \leq 1$ is a parameter controlling the explicity of the collision ($\theta = 0$ means a fully explicit collision whereas $\theta = 1$ gives a fully implicit collision). As pointed out by Guo and Zhao [10], only when $\theta = 0.5$, the collision term reaches the second-order accuracy in time. To eliminate the implicitness of the above scheme, we introduce a new EDF [10]:
\[
g_i = f_i + w_\theta (f_i - f_i^{eq})
\]
with its EDF given by $g_i^{eq} = f_i^{eq}$. Obviously, the macroscopic density and velocity defined from the velocity moments of the distribution $f_i$ can now be calculated from:
\[
\rho = \sum_{i=0}^{8} g_i = \sum_{i=0}^{8} g_i^{eq},
\]
\[
\rho \vec{u} = \sum_{i=0}^{8} \tilde{c}_i g_i = \sum_{i=0}^{8} \tilde{c}_i g_i^{eq}.
\]

With this new EDF, Eq. (37) can be further rewritten as
\[
g_i^{n+1} + \frac{\Delta t}{1 + w_\theta} \left[ \frac{1}{r} c_\varphi \frac{\partial \tilde{g}_i^n}{\partial \varphi} + c_r \frac{\partial \tilde{g}_i^n}{\partial r} \right] = -\frac{1 - w(1-\theta)}{1 + w_\theta} \tilde{g}_i^n + w(1-\theta) f_i^{eq,n} + w_\theta \frac{\tilde{c}_i \cdot \tilde{F}}{c_s^2},
\]
where $w = \Delta t / \tau$ and $\tilde{g}_i = g_i + w_\theta f_i^{eq}$. 

\[\]
We now discuss how to discretize the spatial derivatives in the polar coordinates. In the present study, we use the second-order upwind finite-difference scheme to discretize the derivative in the \( \varphi \)-direction as
\[
\frac{\partial \hat{g}}{\partial \varphi} = \frac{\hat{g}(k, j) - \hat{g}(k - 1, j)}{\Delta \varphi} + \frac{\hat{g}(k, j) - 2\hat{g}(k - 1, j) + \hat{g}(k - 2, j)}{2\Delta \varphi} + \frac{\hat{g}(k, j) - 2\hat{g}(k - 1, j) + \hat{g}(k - 2, j)}{2\Delta \varphi}, \quad (c_{\varphi} > 0),
\]
\[
\frac{\partial \hat{g}}{\partial \varphi} = \frac{\hat{g}(k, j) - \hat{g}(k - 1, j)}{\Delta \varphi} - \frac{\hat{g}(k, j) - 2\hat{g}(k - 1, j) + \hat{g}(k - 2, j)}{2\Delta \varphi} - \frac{\hat{g}(k, j) - 2\hat{g}(k - 1, j) + \hat{g}(k - 2, j)}{2\Delta \varphi}, \quad (c_{\varphi} < 0),
\]
and use a hybrid of the first-order upwind and the second-order central schemes discretize the derivative in the \( r \)-direction as
\[
\frac{\partial \hat{g}}{\partial r} = a\frac{\hat{g}(k, j) - \hat{g}(k, j - 1)}{\Delta r} + (1 - a)\frac{\hat{g}(k, j + 1) - \hat{g}(k, j - 1)}{2\Delta r}, \quad (c_r > 0),
\]
\[
\frac{\partial \hat{g}}{\partial r} = a\frac{\hat{g}(k, j + 1) - \hat{g}(k, j)}{\Delta r} + (1 - a)\frac{\hat{g}(k, j + 1) - \hat{g}(k, j - 1)}{2\Delta r}, \quad (c_r < 0),
\]
where \( k \) and \( j \) are the indices in the angular and radial directions, respectively. With these spatial discretizations, Eq. (41) constitutes an explicit finite difference-base LBGK equation for the macroscopic velocity field.

The discrete velocity BGK equation (29) for the temperature field can also be discretized in the similar manner. Again taking the polar coordinates as an example, the discrete evolution equation (29) can be integrated with respect to time to give
\[
q_{i+1} = q_i + \Delta t\left[\frac{1}{r} c_{\varphi} \frac{\partial q_{i}^{eq}}{\partial \varphi} + c_r \frac{\partial q_{i}^{eq}}{\partial r}\right] = \Delta t\left[\frac{(1 - \theta)}{\tau_i} (q_i - q_{i}^{eq}) - \frac{\theta}{\tau_i} (q_i - q_{i}^{eq})^{n+1}\right].
\]

By introducing the distribution function
\[
q_i = q_i + w_i \theta (q_i - q_{i}^{eq})
\]
and using \( q_{i}^{eq} = \tilde{q}_{i}^{eq} \), we can rewrite Eq. (44) as
\[
q_{i,n+1} = q_i + \Delta t\left[\frac{1}{r} c_{\varphi} \frac{\partial q_{i}^{eq}}{\partial \varphi} + c_r \frac{\partial q_{i}^{eq}}{\partial r}\right] = \Delta t\left[\frac{1 - w_i(1 - \theta)}{1 + w_i \theta} \tilde{q}_{i}^{eq} + w_i (1 - \theta) q_{i}^{eq} \right],
\]
where \( w_i = \Delta t / \tau_i \) and \( \tilde{q}_{i} = q_i + w_i \theta q_{i}^{eq} \). For the convection term, the second-order upwind scheme is employed in both \( \varphi \) and \( r \) directions
\[
\frac{\partial \tilde{q}_{i}}{\partial \varphi} = \frac{\tilde{q}_{i}(k, j) - \tilde{q}_{i}(k - 1, j)}{\Delta \varphi} + \frac{\tilde{q}_{i}(k, j) - 2\tilde{q}_{i}(k - 1, j) + \tilde{q}_{i}(k - 2, j)}{2\Delta \varphi}, \quad (c_{\varphi} > 0),
\]
\[
\frac{\partial \tilde{q}_{i}}{\partial \varphi} = \frac{\tilde{q}_{i}(k, j) - \tilde{q}_{i}(k - 1, j)}{\Delta \varphi} - \frac{\tilde{q}_{i}(k, j) - 2\tilde{q}_{i}(k - 1, j) + \tilde{q}_{i}(k - 2, j)}{2\Delta \varphi} - \frac{\tilde{q}_{i}(k, j) - 2\tilde{q}_{i}(k - 1, j) + \tilde{q}_{i}(k - 2, j)}{2\Delta \varphi}, \quad (c_{\varphi} < 0),
\]
and
\[
\frac{\partial \tilde{q}_{i}}{\partial r} = \frac{\tilde{q}_{i}(k, j) - \tilde{q}_{i}(k, j - 1)}{\Delta r} + \frac{\tilde{q}_{i}(k, j) - 2\tilde{q}_{i}(k, j - 1) + \tilde{q}_{i}(k, j - 2)}{2\Delta r}, \quad (c_r > 0),
\]
\[
\frac{\partial \tilde{q}_{i}}{\partial r} = \frac{\tilde{q}_{i}(k, j + 1) - \tilde{q}_{i}(k, j)}{\Delta r} - \frac{\tilde{q}_{i}(k, j + 1) - 2\tilde{q}_{i}(k, j + 1) + \tilde{q}_{i}(k, j + 2)}{2\Delta r} - \frac{\tilde{q}_{i}(k, j + 1) - 2\tilde{q}_{i}(k, j + 1) + \tilde{q}_{i}(k, j + 2)}{2\Delta r}, \quad (c_r < 0).
\]
The resultant finite difference-based LBGK equation (46) is explicit and can be used to model the temperature field. Note that the temperature is calculated from the new distribution as follows

$$T = \sum_{i=0}^{8} q_i^i,$$  

(49)

It is noted that we can use an average temperature $T_0$ in the density and temperature PDFs given by Eqs. (19) and (31) [17]. In this case, the coupling between the two LBGK equations (41) and (46) can be fulfilled through the forcing term $F_i$ appearing in Eq. (41). For example, in a natural convection problem, $\vec{F}$ is the buoyancy force and related to the temperature through the Boussnique approximation, i.e. $\vec{F} = \bar{g} \beta (T - T_r)$ with $T_r$ being a reference temperature and $\beta$ the thermal expansion coefficient.

4. Numerical results and discussion

In this section, we use the LBGK model presented earlier to simulate the natural convection heat transfer in a horizontal concentric annulus bounded by two cylinders with different temperatures. The reason we chose this problem is because it is a classical heat transfer problem with curved boundaries, and there are ample of experimental and numerical data for this particular problem in the literature that can be used to validate the accuracy of the present LBGK thermal model.

The problem under consideration is sketched in Fig. 1, where $g$ is the gravitation acceleration, $T_h$ and $T_c$ are the constant temperatures of the inner and outer cylinders, respectively, and $T_h > T_c$. In order to match the experimental conditions given in [1], we set the radius ratio of the outer cylinder, $r_o$, to the inner one, $r_i$, to be 2.6 in all of our simulations. The fluid flow through the annulus is driven by the buoyancy force $\vec{F}$.
and heat transfer behavior of this problem is characterized by two non-dimensional parameters, i.e.: the Prandtl number and the Rayleigh number, which are defined as

\[ Pr = \frac{v}{\alpha}, \quad Ra = \frac{g\beta \Delta T L^3}{v\alpha}, \]

where \( \Delta T = T_h - T_c \) is the temperance difference between the hot and cool cylinders, and \( L = r_o - r_i \) is the characteristic length. In our simulations, the macroscopic velocity is non-dimensionalized with a characteristic velocity \( u/L \), and the temperature is non-dimensionalized as \( \vartheta = (T - T_c)/(T_h - T_c) \). As a result, the dimensionless buoyancy force is \( Ra Pr \vartheta \), and the dimensionless viscosity and thermal diffusivity are \( Pr \) and 1.0, respectively. Meanwhile, the average density is set to be 1.0 in our simulations.

The boundary conditions in terms of the dimensionless variables are given as follows:

\[ r^* = \frac{r_o}{L} = 1.625 \quad u^*, v^* = 0, \quad \vartheta = 0, \]

\[ r^* = \frac{r_i}{L} = 0.625 \quad u^*, v^* = 0, \quad \vartheta = 1, \]

\[ \varphi = \pm \frac{\pi}{2} \quad u^* = 0, \quad \frac{\partial v^*}{\partial \varphi} = \frac{\partial \vartheta}{\partial \varphi} = 0 \text{ (symmetry).} \]

The corresponding boundary conditions for the distribution functions are specified according to the following rules:

1. In the \( \varphi \) direction, we use the mirror symmetric boundary treatment \[9\]. Two additional rows indexed by 0 and \( n_x + 1 \) are added. They are symmetric with the first and the last rows in the computation domain with respect to the symmetric line, respectively. The distribution functions on those rows are specified by

\[ g_i(nx + 1, j) = g_i(nx, j) \quad i = 0, 2, 4, \]

\[ g_3(nx + 1, j) = g_1(nx, j), \quad g_1(nx + 1, j) = g_3(nx, j), \]

\[ g_{i+1}(nx + 1, j) = g_i(nx, j) \quad i = 5, 7, \]

\[ g_{i-1}(nx + 1, j) = g_i(nx, j) \quad i = 6, 8, \]

\[ g_i(0, j) = g_i(1, j) \quad i = 0, 2, 4, \]

\[ g_3(0, j) = g_1(1, j), \quad g_1(0, j) = g_3(1, j), \]

\[ g_{i+1}(0, j) = g_i(1, j) \quad i = 5, 7, \]

\[ g_{i-1}(0, j) = g_i(1, j) \quad i = 6, 8. \]

Similarly, such treatments can also be employed to specify the values of \( q' \) at boundary nodes.

2. In the \( r \) direction, we specified the values of \( g_i \) on the boundaries according to the non-equilibrium extrapolation rule \[34\]. In this treatment, the distribution function on each boundary node is decomposed into the equilibrium part and non-equilibrium part. The non-equilibrium part is given by \( g^{neq}(i, ny + 1) = g(i, ny) - g^{eq}(i, ny), \quad g^{neq}(i, 0) = g(i, 1) - g^{eq}(i, 1) \), and the equilibrium part is approximated by,
where for the nodes on \( j = ny + 1 \), \( \rho'(i, ny + 1) = b \rho''(i, ny) + (1 - b) \rho''(i, ny + 1) \) and for the nodes on \( j = 0 \), \( \rho(i, 0) = b \rho''(i, 1) + (1 - b) \rho''(i, 0) \), with the superscript \( n \) denoting the \( n \)th time interval and \( b = 0.05 \). The temperature distribution functions \( q_i \) on the boundary nodes can be treated in a similar manner, except that the exact boundary values of temperature are used in evaluating the equilibrium part of \( q_i \).

In our numerical simulations, the dimensionless relax time \( w \) is taken in the range of 1.2–1.4, and we use different mesh grids for different \( Ra \) numbers. As \( Ra \) is below \( 5 \times 10^4 \), a \( 80 \times 80 \) grid is used, and as \( Ra \) number is beyond this value, a \( 100 \times 100 \) grid is used in order to maintain the accuracy. In our simulations, the parameter \( \theta \) is chosen to be to 0.5 so that the collision term reaches the second-order accuracy in time [10]. Additionally, for the sake of numerical stability, the Courant–Friedricks–Lewey (CFL) number is enforced to be smaller than 1.0 in our simulations. Specifically, we set \( c \Delta t / \Delta r \approx 0.35 \) for all cases, except \( c \Delta t / \Delta r \approx 0.25 \) when \( Ra = 1.02 \times 10^5 \). In evaluating \( \partial g / \partial \phi \) at nodes next to the boundary, (i.e. \( i = 1 \) or \( i = n_x \)), we use the finite difference scheme given by Eq. (43) since the second-order upwind scheme is unavailable. The coefficient \( a \) in Eq. (43) is set to be 0.05, which can effectively reduce the numerical viscosity without scarifying the numerical stability.

The velocity fields and isotherms at the final steady states for different \( Ra \) and \( Pr \) numbers are shown in Fig. 2. For the small \( Ra \) number case, the fluid motion driven by the buoyancy force is very slow, leading to rather weak convection. Consequently, the corresponding isothermals exhibit rather slight difference as compared to those of pure heat conduction between the annulus. On the other hand, when \( Ra \) number increases, the buoyancy force accelerates the circulation of fluid flow and natural convection is significantly enhanced. As a result, the center of the circulation is impelled upward and the corresponding isothermals are greatly deformed. Since \( Ra \) considered in this work is below \( 2.0 \times 10^5 \), oscillating plume and turbulence were not observed in our simulations. The above-discussed behaviors are also found in the previous experimental and numerical studies [1,3].

Additionally, we also calculated the average \( \overline{Nu} \) numbers, which is defined as

\[
\overline{Nu} = \frac{1}{2} (Nu_{inner} + Nu_{outer}).
\]  

The Nusselt numbers obtained in this work, the previous experimental data [1], and the previous numerical results by the finite element method [3] are compared in Table 1. It is seen that the numerical results obtained by the present LBGK thermal model agree well with those reported in
Fig. 2. The velocity fields and isotherms for different $Ra$ numbers and $Pr$ numbers.

Table 1
The average $\bar{Nu}$ number for different $Ra$ numbers and $Pr$ numbers

<table>
<thead>
<tr>
<th>$Ra$</th>
<th>Present work</th>
<th>Experimental [1]</th>
<th>Numerical [3]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.38 \times 10^3$, $Pr = 0.716$</td>
<td>1.320</td>
<td>1.38</td>
<td>_a</td>
</tr>
<tr>
<td>$9.50 \times 10^3$, $Pr = 0.717$</td>
<td>1.999</td>
<td>2.01</td>
<td>1.9901</td>
</tr>
<tr>
<td>$3.20 \times 10^4$, $Pr = 0.717$</td>
<td>2.911</td>
<td>2.89</td>
<td>_a</td>
</tr>
<tr>
<td>$6.19 \times 10^4$, $Pr = 0.718$</td>
<td>3.361</td>
<td>3.32</td>
<td>3.3092</td>
</tr>
<tr>
<td>$1.02 \times 10^5$, $Pr = 0.718$</td>
<td>3.531</td>
<td>3.66</td>
<td>3.6475</td>
</tr>
</tbody>
</table>

\(^a\) \(^-\) denotes the cases which are unavailable in [3].
the previous studies for the $Ra$ and $Pr$ numbers considered here. The relative errors in all the cases considered are all within 5%. This demonstrated that the present LBGK model for thermal flows predicts accurately the natural convective heat transfer in a horizontal concentric annulus.

It is worth mentioning that the authors also simulated this problem using the FDLBGK model proposed by Mei and Shyy [9] under the same conditions as discussed above, including the boundary treatments, the difference schemes, the value of $w$ and the CFL number. It is found that for this particular problem, this FDLBGK model becomes unstable when $a$, the coefficient in Eq. (43), is equal to or below 0.5. It is noted that $a$ in Eq. (43) actually represents the proportion of the first-order upwind scheme in the hybrid scheme. When $a$ is below 0.5, the effect of the central difference scheme becomes dominant, and this would lead to some instability. On the other hand, when $a$ is chosen to be above 0.5, the first-order upwind scheme is dominant and will bring about a significant numerical viscosity. For example, as $Ra = 9.5 \times 10^3$, the average $Nu$ number is merely 1.67 for $a = 0.5$, deviating greatly from the experimental result $Nu = 2.01$.

5. Concluding remarks

In this paper, we have presented a finite difference-based LBGK model for thermal flows by discretizing two discrete velocity Boltzmann equations for the density and temperature distribution functions which originate from the continuous Boltzmann equation. The method is fully explicit and exhibits good numerical stability. This model has been applied to simulate the natural convection heat transfer in a concentric horizontal annulus. The numerical results agree well with the experimental and numerical results reported in the previous studies [1,3].

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References