Determining the up-down-up response through tension tests of a pre-twisted shape memory alloy tube

Pingping Zhu a, Ping Feng b, Qing-Ping Sun c, Jiong Wang d, Hui-Hui Dai a,*

a Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon Tong, Hong Kong
b College of Mechatronics and Control Engineering, Shenzhen University, Shenzhen, 518060, China
c Department of Mechanical Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong
d Department of Mechanics, South China University of Technology, 381 Wushan Road, Tianhe District, Guangzhou, 510641, China

Abstract

In this paper, the stress-induced phase transitions in a shape memory alloy (SMA) tube under torsion and pre-twisted tension are studied analytically. A constitutive model with the specific forms of Helmholtz free energy and mechanical dissipation rate is employed to formulate the governing system. Exact solution of the tube under pure torsion is first derived and the shear stress—shear strain response is determined, which reveals the hardening effect. For the pre-twisted tube under uniaxial tension, the one-dimensional asymptotic tensile stress—tensile strain relations for the austenite, the phase transition and the martensite regions are derived by using the asymptotic expansion method. By properly defining an elastic energy potential, the present system can be viewed as an elastic problem, which can be related to the problem of Ericksen’s bar. The analytical formulas for the nucleation and propagation stresses in terms of the pre-shear strain (caused by the pre-twist) are obtained. Tension tests with fixed pre-twists on SMA thin-walled tube are conducted, with a focus on the stress strain response. The measured values and the analytical formula for the propagation stress are used to determine the material parameters, which, in turn, yields the up-down-up response of a shape memory alloy tube under pure tension. The tendency and turning point of the phase transformation in the pre-twisted tube from localization to homogeneous deformation are also determined, which suggests a plausible way to avoid the instability in actuation applications.

© 2016 Elsevier Ltd. All rights reserved.

1. Introduction

Shape memory alloys (SMAs) have been successfully applied in a variety of fields, including medicine, aerospace and automation (see Jani et al., 2014). The thermo-mechanical behaviors of this group of materials exhibit two remarkable properties: the shape memory effect (SME) and the pseudo-elasticity (PE). The SME means that at low temperatures SMAs can be deformed under external loads and the deformations can be recovered upon heating above the transition temperature. The PE means that at appropriate (high) temperatures, SMAs are able to recover their undeformed state if the applied external forces are removed. The underlying mechanisms of these two unusual properties are the stress- or temperature-induced phase transitions between the austenite (A) phase and the (twinned or detwinned) martensite (M) phase.
In order to gain a deep understanding of the thermo-mechanical response of SMAs, a series of experiments, including the uniaxial and the biaxial loading tests, have been conducted on SMA specimens with different geometries. In Li and Sun (2002), He and Sun (2009), Zhou and Sun (2011), the behaviors of SMA thin-walled tubes under various loading conditions have been systematically investigated. In the pure tension tests, it was found that the deformation of the tubes can be inhomogeneous, which is macroscopically reflected in the nucleation and propagation of martensite/austenite bands with sharp A-M interfaces. The stress–strain curve during a whole tensile loading cycle forms a hysteresis loop, which implies that energy dissipation occurs during the phase transition process. On the other hand, the responses of SMA thin-walled tubes under pure torsion or combined tension–torsion loading conditions have also been systematically investigated (see Lim and McDowell, 1999; Sun and Li, 2002; McNaney et al., 2003; Mehrabi et al., 2012; Wang et al., 2012; Mehrabi and Kadkhodaei, 2013; Mehrabi et al., 2015a,b). In Sun and Li (2002), it was found that the deformation of the tube under pure torsion is axially homogeneous and the recorded stress–strain curves for both the loading and unloading processes represent monotonic hardening. In the tension tests of a pre-twisted SMA tube, it was found that the deformation of the tube changes gradually from the localization form to the homogeneous form with the level of the pre-twisted angle gradually increasing. The recorded tensile stress–tensile strain curve becomes monotonically hardened. The von-Mises circle plotted in Sun and Li (2002) and the relevant surface morphology observation showed that the tube has a strong anisotropic response during stress-induced transformations. Significant differences of the responses of SMA specimens in torsion and tension have also been reported in Wang et al. (2012), Mehrabi and Kadkhodaei (2013), Mehrabi et al. (2015a,b). Among these works, Mehrabi et al. (2015b) investigated the responses of NiTi thin-walled tubes under proportional and non-proportional tension–torsion tests based on stress-control or strain-control loading patterns. They also found the anisotropic response of the NiTi tubes in phase transitions.

For the instability phenomena of SMAs observed in the tension tests, the localization configuration of the tube is unfavorable for some structural and actuation applications. Based on this consideration and the above mentioned experimental investigations, two issues are of interest in our present work.

First, studies have focused on finding the true material response over the Lüders-like stress plateau in literature (see Shaw and Kyriakides, 1997; Shaw and Kyriakides, 1998; Li and Sun, 2002). However, it is rather difficult to calibrate material parameters in many proposed constitutive models and to deduce the softening part of the stress–strain curve. Some works have subtly overcome this difficulty. For example, Hallai and Kyriakides (2013) produced the up-down-up stress–strain response of a NiTi strip by using an experimental technique that was first reported in Shioya and Shiroiri (1976). They conducted uniaxial tension tests on a laminate (consisting of two stainless steel strips as the face-strips and one NiTi layer as the core) and the stainless steel, respectively. The stainless steel is known to be a hardening material while NiTi is an unstable material. Under proper design, the instability of the core in such laminates was found to be suppressed by the hardening effect of the face-strips. This was structurally reflected in the recorded monotonic hardening stress–strain curves of the laminate. Subsequently, they extracted the material response of NiTi from the response of the laminate and the face-strips, which was found to have an up-down-up fashion. Moreover, Song et al. (2013) proposed an analytical approach based on the constitutive model proposed in Rajagopal and Srinivasa (1999). They adopted specific forms of the Helmholtz free energy and the rate of mechanical dissipation to study phase transformations of a NiTi wire under uniaxial tension. By taking advantage of the small strains of SMAs, they used the asymptotic expansion method developed in some previous works (see Cai and Dai, 2006; Dai and Cai, 2006; Dai and Wang, 2009, 2010; Wang and Dai, 2010; Wang and Dai, 2012a,b) to tackle the problem analytically. By further relating the problem to the equilibrium theory of Ericksen (1975), they deduced the analytical formulas of the nucleation and propagation stresses, which were used to calibrate material constants by comparing with the relevant experimental data. Consequently, the nominal stress–strain curve was captured, which also shows an up-down-up fashion. One deficiency of this asymptotic analytical study is that determining the material constants requires nucleation stresses. Generally speaking, the value of the nucleation stress is difficult to be measured in the experiments. On the other hand, the propagation stress can be clearly read out from the measured stress–strain curves. In this regard, we aim at providing another analytical approach to capture the material response of SMAs by using the propagation stress measured in the experiments.

More specifically, we study the stress-induced phase transitions in an SMA thin-walled tube based on a three-dimensional setting. Homogeneous and piecewise homogeneous deformations of such a tube under pre-twisted tension are considered. We aim at obtaining the analytical formulas of the nucleation and propagation stresses in terms of the material constants and the pre-shear strain (induced by the pre-twist). In order to determine the material constants, a series of tension tests with fixed pre-twists on SMA thin-walled tube are conducted, with a focus on the stress–strain response. The measured values and the analytical formula for the propagation stress are then used to determine the material constants. Subsequently, an up-down-up tensile stress–tensile strain curve is captured.

Another motivation of the current work comes from some previous efforts made to widen the controllable range of SMAs by creating transformation loading gradient along the deforming direction (see Mahmud et al., 2007; Mahmud et al., 2008; Shariat et al., 2012, 2013). Geometrically graded and microstructurally graded NiTi alloys are usually used to achieve this goal. Noticing the undesirable effects on some actuation applications the material instability of SMAs has, Shariat et al. (2012) tapered a NiTi bar to provide a gradually changing cross-sectional area so that the uniaxial tensile stress varies along the length of the bar. This treatment geometrically created a transformation stress gradient along the length of the bar,
which makes the uniaxial tensile test of the tapered bar behave stably and results in a monotonic hardening stress—strain curve. Later, Shariat et al. (2013) provided another approach through designed gradient anneal of the sample, which can obtain a microstructurally graded NiTi wire. Such a wire has continuously varying transformation stress and strain along its length. Different experiments were designed and conducted, and it was found that the slopes of the stress—strain curves in the phase transition regions are positive and increase with the temperature range of the gradient anneal increasing. Inspired by these results, we also aim at providing an alternative method to harden the slope of the response in the phase transition region.

To achieve this goal, homogeneous deformations of the SMA thin-walled tube under pure torsion are analytically studied to determine the hardening effect of torsion. Then we analyze the influence of pre-twist on the subsequent phase transitions under tension. Based on the derived analytical formulas for the nucleation and propagation stresses in terms of the pre-shear strain and the determined material parameters, we quantitatively predict the tendency of the stress drop (i.e., difference of the nucleation stress and the propagation stress) shown in the tensile stress—tensile strain curves for different pre-strains. Furthermore, the critical value of the pre-shear strain at which the deformations under tension become homogeneous is determined. Thus if the given pre-shear strain is higher than this critical value, the tensile stress—tensile strain curve becomes monotonically hardened.

In order to derive the analytical solutions, a proper constitutive model is necessary. Up to now, a large number of constitutive models (see Cisse et al., 2016) have been proposed to describe the thermo-mechanical behavior of polycrystalline SMAs, mainly including the micromechanics based models (see Cherkasov et al., 1998; Levitas and Ozsoy, 2009a,b; Yu et al., 2013) and the macroscopic phenomenological models (see Leclercq and Lexcellent, 1996; Rajagopal and Srinivasa, 1999; Zaki and Moumni, 2007; Arghavani et al., 2010a,b; Morin et al., 2011; Lagoudas et al., 2012; Teeriahm, 2013; Yu et al., 2015). Generally speaking, micromechanics based models take into account the influence of microstructure on the macroscopic response of SMAs through homogenization techniques. Investigation of the microstructure of SMAs can be of great help to understand the macroscopic phenomena. However, it often requires a lot of computational efforts to evaluate the macroscopic structural components. On the other hand, macroscopic phenomenological models study the bulk behavior of SMAs at the macroscopic scale, which often incorporate internal state variables (e.g., martensite volume fraction) into the macroscopic energy potentials. Many resulting constitutive systems are very complex, such that numerical methods are required to implement them (see Müller and Bruhns, 2006; Moumni et al., 2008; Lagoudas et al., 2012; Sedlák et al., 2012). These numerical simulations often have difficulty in choosing the values for material parameters. In this aspect, some existing analytical studies (see Ericksen, 1975; Abeyaratne and Knowles, 1993; Levitas et al., 2010) show multiple advantages. For example, the analytical results can clearly manifest the effects of the state variables and can be used to calibrate the material constants and predict the material responses by combining with the experimental data.

In this paper, we employ the constitutive theory proposed in Rajagopal and Srinivasa (1999) to model the thermo-mechanical behavior of the SMA tube. A key point of this constitutive formulation is the concept of multiple natural configurations (see Rajagopal and Srinivasa (1998)), which is related to a phase state variable. Although this model is based on one-dimensional setting, it is easy to extend it to three dimensions since their fundamental concepts are generalizable to three dimensions. Besides that, the rate-independent setting is convenient for studying a quasi-static process and deriving analytical solutions. Specific forms of the Helmholtz free potential and rate-independent dissipation mechanisms are adopted to complete the formulation. The evolution of the phase state is considered and the one-dimensional governing equations for pure torsion and pre-twisted tension conditions are obtained. For the tube under pure torsion, the shear stress—shear strain relations for the three phase regions are successfully derived. By comparing with the experimental data of pure torsion test in Sun and Li (2002), the hardening material response of the tube under pure torsion is determined, which reveals the hardening effect of pre-twist on the SMA tube. For the pre-twisted tube under uniaxial tension, by taking advantage of the small values of strains the asymptotic expansion method is used to derive the asymptotic nominal stress—strain relations corresponding to the three phase regions. The resulting tensile stress—tensile strain curve turns out to be still non-monotonically increasing. Thus we associate both the tensile loading and tensile unloading processes to the equilibrium theory of the Ericksen’s bar (see Ericksen, 1975). Based on the results of Ericksen (1975), we derive analytical formulas for the nucleation and propagation stresses in terms of material parameters and the pre-shear strain. Although Sun and Li (2002) have conducted pre-twisted tension tests on SMA tubes, the available data on propagation stress are not enough to determine the relevant material constants. So we also conduct tension tests on the pre-twisted SMA tube and sufficient data on the propagation stress are measured. Then the material constants are determined by comparing the measured values and the analytical formula for the propagation stress. Consequently, the up-down-up material stress—strain relation of the tube under pure tension is successfully captured (which is achieved by setting the pre-shear strain to be zero in the resulting tensile stress—tensile strain curve). The tendency of the stress drop of the pre-twisted tube under tension with varying pre-shear strains is also predicted. In addition, the critical value of the pre-shear strain for the homogeneous deformation is determined. Thus one way to avoid instability in actuation applications by using SMA tubes is suggested: If the tube is pre-loaded by a shear strain (higher than a critical value), the response of the pre-twisted tube under tension is stable. As a repeated validation of the analytical results and the methodology of the present work, the experimental results of the tension tests with fixed pre-twists in Sun and Li (2002) are also used to determine the material response.
The main goal of the present work is to determine the up-down-up material response under pure tension and the turning point of the phase transformation in the pre-twisted tube from localization form to homogeneous form. The novelty is that by an asymptotic method and the theory of Ericksen’s bar, analytical formulas for the nucleation and propagation stresses in terms of the pre-shear strain are derived. Then, by comparing with the measurements of the conducted experiments, we achieve the main goal. It appears that there still lacks a systematical way to calibrate material parameters in constitutive models of SMAs. Here, the derived analytical formula for the propagation stress under different pre-shear strains, which can be measured with little error in experiments, can be used to determine the material parameters.

This paper is organized as follows. In Section 2, the constitutive model proposed in Rajagopal and Srinivasa (1999) is briefly introduced. In Section 3, we formulate the field equations for a transversely isotropic SMA thin-walled tube. In Section 4, the homogeneous deformation of the tube under pure torsion is studied, which yields the exact shear stress—shear strain relation. Then, in Section 5, we focus on the homogeneous and piecewise homogeneous deformations of the pre-twisted SMA thin-walled tube under tension. Subsequently, in Section 6, we use the asymptotic expansion method and the phase transition criteria to derive the one-dimensional tensile stress—tensile strain relation for three regions in terms of the pre-shear strain. In Section 7, the present system is related to the Ericksen’s bar problem, and the analytical formulas for the nucleation and propagation stresses in terms of the pre-shear strain are obtained. In Section 8, the experimental results of the tension tests on the pre-twisted SMA tubes are first reported. Then the experimental data are used to determine the material parameters, which, in turn, yields the up-down-up response in the pure tension test. It is also found that by increasing the pre-twist to a certain value a better controllability in actuation applications can be achieved. Moreover, as a repeated validation, the experimental results of the tension tests with fixed pre-twists in Sun and Li (2002) are also applied to determine the material response. Finally, some conclusions are drawn in Section 9.

2. Constitutive model

Our present study is carried out based on the constitutive model with a Helmholtz free energy $\Phi$ proposed in Rajagopal and Srinivasa (1999). Here, we give a brief introduction of this model.

First, starting from the thermodynamic laws and through some conventional derivations (see Song (2013)), the following constitutive equations can be derived

$$ S = \frac{\partial \Phi(F, \alpha, T)}{\partial F}, \quad \eta = -\frac{\partial \Phi(F, \alpha, T)}{\partial T}, \quad \xi = -\frac{\partial \Phi(F, \alpha, T)}{\partial \alpha}, $$

where $F$ is the deformation gradient from the reference configuration $k_0$ to the current configuration $k_t$, $S$ is the nominal stress tensor, $\eta$ is the entropy per unit referential volume, $\xi$ is the rate of mechanical dissipation, $T$ is the absolute temperature and $\alpha$ is the phase state variable representing the volume fraction of the martensite phase. To complete this constitutive system, the constitutive forms of the energy function $\Phi$ and the dissipation rate $\xi$ need to be proposed. For this purpose, we mention the concept of multiple natural configurations proposed by Rajagopal and Srinivasa (1998, 1999).

Denote $k_0$ and $k_1$ as the unstressed austenite configuration (the reference configuration) and the martensite configuration, respectively. The deformation gradient tensor corresponding to the mapping from $k_0$ to $k_1$ is denoted as $G$. For any given phase state value $\alpha$, it is supposed that there exists an intermediate natural configuration $k_\alpha$. The deformation gradient tensor $G_\alpha$ corresponding to the mapping from $k_0$ to $k_\alpha$ is then defined by

$$ G_\alpha := (1 - \alpha)I + \alpha G. $$

Let $F_\alpha$ be the deformation gradient tensor from $k_\alpha$ to $k_t$. Then, the chain rule yields that

$$ F_\alpha = FG_\alpha^{-1}. $$

Based on the concept of multiple natural configurations, the following constitutive form of Helmholtz free energy per unit reference volume has been proposed in Rajagopal and Srinivasa (1999), which has also been adopted in Wang and Dai (2012a) and Song et al. (2013).

$$ \tilde{\Phi}(F_\alpha, \alpha, T) = \det(G_\alpha)[(1 - \alpha)\Phi_1(F_\alpha) + \alpha\Phi_2(F_\alpha)] + B\alpha(1 - \alpha) + [(1 - \alpha)\phi_1(T) + \alpha\phi_2(T)], $$

where $\Phi_1$ and $\Phi_2$ are the strain energy functions of the austenite and the martensite phases respectively, the second term represents the interfacial energy with the interfacial constant $B$, $\phi_1$ and $\phi_2$ are the thermal free energies of the austenite and the martensite phases respectively. More detailed discussion on the choice of the interfacial energy can be found in Müller (1989). By substituting (4) into (1), the expression of the nominal stress $S$ can be derived.
For the rate of mechanical dissipation $\xi$, the following constitutive form was proposed in Rajagopal and Srinivasa (1999)

$$
\xi = \begin{cases} 
A^+(\alpha) |\dot{\alpha}| & \text{if } \dot{\alpha} > 0, \\
A^-(\alpha) |\dot{\alpha}| & \text{if } \dot{\alpha} \leq 0.
\end{cases}
$$

(5)

Here, $A^+(\alpha) \geq 0$ and $A^-(\alpha) \geq 0$ are the forward and backward dissipative resistances, which are assumed to have the following specific forms

$$
A^+(\alpha) = k^+ \alpha + Y^+, \quad A^-(\alpha) = k^- (1 - \alpha) + Y^-,
$$

(6)

where $k^+$ and $Y^+$ are the dissipative constants. These constitutive forms have also been adopted by Song et al. (2013), which are more general than those adopted in Rajagopal and Srinivasa (1999) and Mirzaeifar et al. (2011).

For future convenience, we further define the following “driving force for phase transformation”

$$
D(F, \alpha, T) := -\frac{\partial \Phi}{\partial \alpha}.
$$

(7)

Thus the dissipation equation (1)3 can be written as $\xi = -D \dot{\alpha}$. Together with (5), we arrive at the following phase transition criteria

$$
\begin{align*}
- A^- (\alpha) < D < A^+ (\alpha) & \Rightarrow \dot{\alpha} = 0, \\
\dot{\alpha} \neq 0 & \Rightarrow D = \begin{cases} 
A^+(\alpha) & \text{if } \dot{\alpha} > 0, \\
A^-(\alpha) & \text{if } \dot{\alpha} < 0.
\end{cases}
\end{align*}
$$

(8)

In view of (8), it can be found that if $D$ lies between $-A^- (\alpha)$ and $A^+ (\alpha)$, the response of the material is elastic without any phase state changing, while if $\dot{\alpha} \neq 0$, phase transition happens and $D$ must be equal to one of the dissipative resistances. This indicates that the expression of the phase state $\alpha$ in phase transition region ($0 < \alpha < 1$) can be derived by setting $D = A^+ (\alpha)$ or $D = -A^- (\alpha)$.

3. Field equations of a transversely isotropic SMA thin-walled tube

Now we consider an SMA circular cylindrical thin-walled tube with inner radius $A$, outer radius $B$ and length $L$. Suppose that the tube is homogeneous, stress-free and entirely made up of austenite phase in its reference configuration $\kappa_0$. The thickness of the tube is assumed to be very small such that the radius $R$ in the reference configuration can be regarded as $(A + B)/2$. We shall consider isothermal and static deformations of this tube, then the nominal stress tensor satisfies the following field equations (neglecting the body forces)

$$
\text{Div}(\mathbf{S}) = 0,
$$

(9)

where Div represents the divergence in the reference configuration. At present, we only consider homogeneous or piecewise homogeneous deformations of the tube. Therefore the above field equations are automatically satisfied.

Notice that such an SMA tube behaves anisotropically when phase transition happens (see Sun and Li, 2002; Mehrabi et al., 2015b), thus we further assume the tube is transversely isotropic elastic in both the austenite and the martensite phases. Furthermore, we assume that the elastic strains of the SMA tube in both the austenite and the martensite phases are relatively small. Thus, within the framework of linear elasticity, the stress–strain relation has the following form

$$
\sigma = \mathbf{C} : \epsilon,
$$

(10)

where $\sigma$ is the stress tensor, $\epsilon$ is the infinitesimal strain tensor defined by $\epsilon := [(\mathbf{F} - \mathbf{I}) + (\mathbf{F} - \mathbf{I})^T]/2$, and $\mathbf{C}$ is the fourth-order stiffness tensor. For transversely isotropic material, $\mathbf{C}$ has only five independent material constants.

Let $\mathbf{C}^0 (i = 1, 2)$ be the stiffness tensors of the austenite and martensite phases respectively, and

$$
\epsilon_{\alpha} := \frac{1}{2} [(\mathbf{F}_{\alpha} - \mathbf{I}) + (\mathbf{F}_{\alpha} - \mathbf{I})^T]
$$

(11)

be the infinitesimal strain tensor corresponding to the mapping from $\kappa_\alpha$ to $\kappa_i$. Correspondingly, the strain energy functions take the following forms

$$
\Phi_i (\epsilon_{\alpha}) = \frac{1}{2} (\mathbf{C}^0_i : \epsilon_{\alpha}) : \epsilon_{\alpha}, i = 1, 2.
$$

(12)

By substituting (4) and (12) into (1)1, the nominal stress tensor can be derived as
\[ S = \frac{\partial \Phi}{\partial F} = \det(G_a)G_a^{-1}[(1 - \alpha)\frac{\partial \Phi_1(F_a)}{\partial F_a} + \alpha \frac{\partial \Phi_2(F_a)}{\partial F_a}] = \det(G_a)G_a^{-1}[(1 - \alpha)\sigma^{(1)} + \alpha \sigma^{(2)}], \]  

where \( \sigma^{(i)} := C^{(i)} : \epsilon_a. \)

The cylindrical coordinate system is used to describe the mechanical system in the following sections. With the aim of revealing the hardening effect of torsion on the response of SMA tubes, we first consider the deformations of such a tube under pure torsion.

4. An analytical model for pure torsion

First, we consider an SMA tube subjected to a torsional force (see Fig. 1). Denote \((R, \Theta, Z)\) and \((r, \theta, z)\) as the coordinates of a point of the tube in the reference and current configurations, respectively. The homogeneous deformation under pure torsion is defined by

\[ r = R, \quad \theta = \Theta + \frac{\Theta_0}{L}Z, \quad z = Z, \]  

where \(\Theta_0\) is the rotational angle of the tube. Then, the corresponding deformation gradient tensor \(F\) is given by

\[
(F) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & M \\
0 & 0 & 1
\end{pmatrix},
\]  

where the shear strain \(M := \Theta_0 R / L = \Theta_0 (A + B) / (2L)\). For such a twisting deformation, the deformation gradient tensor \(G\) corresponding to the mapping from \(k_0\) to \(k_1\) is taken to be

\[
(G) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & s_0 \\
0 & 0 & 1
\end{pmatrix},
\]  

where \(s_0\) is the transformation strain along the shear direction, which satisfies \(0 < s_0 < 1\). Substituting (16) into (2), we can obtain the deformation gradient tensor \(G_a\) corresponding to the mapping from \(k_0\) to \(k_a\). From (3) and (11), we further get the corresponding infinitesimal strain tensor

\[
(\epsilon_a) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \frac{1}{2}(M - \alpha s_0) \\
0 & \frac{1}{2}(M - \alpha s_0) & 0
\end{pmatrix}.
\]  

Using the Voigt notation, the relation \(\sigma^{(i)} := C^{(i)} : \epsilon_a\) can be rewritten in another form

\[ \text{Fig. 1. The tube under pure torsion.} \]
In the framework of linear elasticity, the assumption of small strains in both the austenite phase and the martensite phase implies that the deformation gradient is close to unit tensor $I$. So the current configuration can be regarded to be the same as the reference configuration in either the austenite or the martensite phases. Thus, the subscripts of $\sigma^{(i)}$ are all written in the general form of lower case. It can be seen that there is only one nonzero component of $\sigma^{(i)}$

$$\sigma^{(i)}_{\theta z} = C^{(i)}_{44}(M - \alpha s_0), \ i = 1, 2. \tag{19}$$

From (12), we have

$$\Phi_t = \frac{1}{2} C^{(i)}_{44}(M - \alpha s_0)^2, \tag{20}$$

and further from (13), we obtain the only one nonzero component of the nominal stress tensor

$$S_{\theta z} = [(1 - \alpha)C^{(1)}_{44} + \alpha C^{(2)}_{44}](M - \alpha s_0). \tag{21}$$

Suppose the tube is subjected to the torsional couple $T$. Then, on each cross section of the tube, we have

$$T = \frac{2\pi}{\alpha} \int_0^B S_{\theta z} R^2 dR d\theta = \frac{2\pi}{3} \left( B^3 - A^3 \right) S_{\theta z} \Leftrightarrow S_{\theta z}(M, \alpha) = \tau := \frac{3T}{2\pi(B^3 - A^3)}, \tag{22}$$

where $\tau$ is the shear stress. Thus by using (21), we get the following one-dimensional shear stress—shear strain relation

$$\tau = [(1 - \alpha)C^{(1)}_{44} + \alpha C^{(2)}_{44}](M - \alpha s_0). \tag{23}$$

In order to get the explicit shear stress—shear strain relations for the austenite, the martensite and the phase transition regions, we need to determine the relation between the phase state variable $\alpha$ and the shear strain $M$. By setting $\alpha = 0$, 1 in (23), we can easily obtain the shear stress—shear strain relations for the austenite and the martensite phases. However, the phase state $\alpha$ varies with $M$ during the phase transition processes. Next we take the loading process as an example to show how to obtain the expression of $\alpha$ by using the phase transition criteria (8).

The response of the tube is initially elastic upon loading with $\alpha = 0$. When phase transition happens, we have $\dot{\alpha} > 0$. From (8), the value of $\alpha$ can be determined by the relation $D = A^+(\alpha)$. When $\alpha$ reaches 1, the response of the tube becomes elastic again and $\alpha$ is unchanged upon further loading. When $0 < \alpha < 1$, by using (4), (7), and (20), we obtain

$$D = \alpha^2 \frac{1}{2} \left( C^{(1)}_{44} - C^{(2)}_{44} \right) s_0^2 + \alpha \left[ 2B - \frac{1}{2} \left( C^{(1)}_{44} \right) s_0^2 - \frac{1}{2} \left( C^{(1)}_{44} - C^{(2)}_{44} \right) M s_0^2 \right] + \frac{1}{2} \left( C^{(1)}_{44} - C^{(2)}_{44} \right) M s_0 - B + \phi_1 - \phi_2. \tag{24}$$

Substituting (6) and (24) into the equality $D = A^+(\alpha)$, we get the relation between $\alpha$ and $M$,

$$\Delta s_0^2 \alpha^2 + \left( 2D^+_{\phi} - \Delta Ms_0 - s_0^2 \right) \alpha - 2D^+_{\phi} + Ms_0 = 0. \tag{25}$$

where

$$\Delta = \frac{C^{(1)}_{44} - C^{(2)}_{44}}{C^{(1)}_{44}}, \ D^+_{\phi} = \frac{B + \phi_2 - \phi_1 + Y^+}{C^{(1)}_{44}} \frac{1}{C^{(1)}_{44}}, \ D^+_k = \frac{2B - k^+}{C^{(1)}_{44}}. \tag{26}$$
By solving this quadratic equation for $\alpha$ and choosing the reasonable root, the exact solution of $\alpha$ can be obtained. Besides that, the critical shear strain values $M_0^+$ and $M_1^+$ (where the subscripts 0,1 denote the values of $\alpha$ at those points) for the start and end of phase transition can also be determined. In fact, these critical strain values can be naturally determined by setting $\alpha = 0, 1$ in (25), which yields that

$$M_0^+ = \frac{2D_0^+}{s_0}, \quad M_1^+ = s_0 \pm \frac{2(D_0^+ - D_1^+)}{(1 - \Lambda)s_0}. \quad (27)$$

For the unloading process, by using $D = -A^- (\alpha)$, we can also obtain the expression of $\alpha$. Finally, we arrive at the following expressions of $\alpha$ for both loading and unloading processes

$$\alpha^\pm = \begin{cases} 0 & \text{if } M < M_0^+, \\ \frac{\Delta M s_0 + s_0^2 - 2D_k^\pm - \sqrt{(\Delta M s_0 + s_0^2 - 2D_k^\pm)^2 - 4\Delta s_0^2 (M s_0 - 2D_k^\pm)}}{2\Delta s_0^4} & \text{if } M_0^+ \leq M \leq M_1^+, \\ 1 & \text{if } M > M_1^+, \end{cases} \quad (28)$$

where we have adopted the notations $D_k^\pm = (B + \phi_2 - \phi_1 - k^- - Y^-)/C_{44}^{(1)}$ and $D_k^\pm = (2B - k^-)/C_{44}^{(1)}$. By substituting (28) into (23), we get the exact shear stress–shear strain relations corresponding to the austenite, the phase transition and the martensite regions

$$\tau = \begin{cases} C_{44}^{(1)} M & \text{if } 0 \leq M < M_0^+, \\ C_{44}^{(1)} \left( B_0^+ + B_1^+ M + \frac{D_k^+}{s_0} \sqrt{B_2^+ \frac{s_0}{B_1^+} + B_3^+ M + B_4^+ M^2} \right) & \text{if } M_0^+ \leq M \leq M_1^+, \\ C_{44}^{(1)} (1 - \Lambda)(M - s_0) & \text{if } M > M_1^+, \end{cases} \quad (29)$$

where the expressions of the material constants $B_0^+, B_1^+, B_2^+, B_3^+, B_4^+$ are given in the Appendix. It is easy to see that the shear stress $\tau$ is an odd function of the shear strain $M$, which should be the case in reality. In fact, if we twist the tube in the opposite direction (i.e. accordingly, $M$ and $s_0$ should be negative), then by the above relation, $\tau$ is also negative.

It can be seen that there are seven independent parameters $C_{44}^{(1)}, s_0, \Delta, D_0^+ \text{ and } D_1^+$ in the shear stress–shear strain relation (29). In order to determine these material parameters, we shall use the experimental results presented in Fig. 6(a) of Sun and Li (2002) (see Fig. 2(a)), which shows the measured shear stress–shear strain curve of a thin–walled Ti-56wt.% Ni tube under pure torsion. From Fig. 2(a), we can obtain four critical strain values and two critical stress values, which are listed in Table 1. Here, the two stresses $\tau_0^+$ and $\tau_1^+$ correspond to the start and end of the phase transition from austenite to martensite (i.e., corresponding to $M_0^+$ and $M_1^+$). As the deformation of the tube under pure torsion is stable, the values we obtained here are quite accurate. Then, $M_1^+ - M_0^+$ yields the transformation strain $s_0$, and $\tau_0^+ / M_0^+$ gives the shear modulus $C_{44}^{(1)}$. Subsequently, the analytical formula for $\tau_1^+$ at $M_1^+$ yields $\Delta$, and the analytical formulas for $M_0^+$ and $M_1^+$ yield $D_0^+$ and $D_1^+$. In Table 2, the determined values of the seven material parameters are listed. Thus by using the analytical formulas (29), the shear stress–shear strain relations for the loading and the unloading processes are determined as follows:

![Shear stress vs. shear strain](image_url)
The shear stress and shear strain data taken from Fig. 6(a) in Sun and Li (2002).

<table>
<thead>
<tr>
<th>$\tau_0$ (MPa)</th>
<th>$\tau_1$ (MPa)</th>
<th>$\tau_2$ (MPa)</th>
<th>$\tau_3$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0166</td>
<td>0.025</td>
<td>373</td>
<td>0.0114</td>
</tr>
</tbody>
</table>

\[
\tau^+ = \begin{cases} 18819M \\ 52.49 + 4160.66M + 25.53\sqrt{27.22 + 1282.82M + 26556M^2} \\ -188.75 + 22469.89M \end{cases} \quad \text{if } 0 \leq M < 0.0166, \\
\tau^- = \begin{cases} 18819M \\ -27.99 + 3947.3M + 24.22\sqrt{47.87 + 1334.97M + 26556M^2} \\ -188.75 + 22469.89M \end{cases} \quad \text{if } 0 \leq M < 0.0114,
\]

where $\tau^\pm$ respectively represent the shear stress for the loading and the unloading processes. It can be easily checked that $\frac{\partial \tau^\pm}{\partial M} > 0$, which analytically verifies the hardening response of the tube under pure torsion. Correspondingly, the shear stress–shear strain curve for the whole torsional loading cycle is plotted in Fig. 2(b), which behaves in a quite consistent pattern with the measured one. Fig. 2(b) also verifies the hardening response in the phase transition regions. These results analytically and quantitatively reveal the hardening effect of torsion on the response of the SMA tubes.

5. Formulation for homogeneous and piecewise homogeneous deformations of a pre-twisted SMA tube under tension

In this section, we concentrate on the deformations of the pre-twisted tube under tension (see Fig. 3). One end of the tube is fixed and the other is pre-loaded by a constant torsional force. A uniaxial tensile force is then applied on this pre-twisted tube along the axial direction. The applied torsional force is kept fixed during the whole tensile loading and unloading processes. The corresponding pre-shear stress $\tau^0$ is set to be lower than the nucleation stress of the tube under pure torsion such that the tube still stays in austenite phase at the start of the tensile loading.

Let $\kappa_0$ be the configuration of the pre-twisted tube before tensile loading (see Fig. 4). Denote $(R, \Theta, Z)$, $(R^*, \Theta^*, Z^*)$, and $(r, \theta, z)$ as the coordinates of a point of the tube in the reference configuration $\kappa_0$, the pre-twisted configuration $\kappa_0^*$, and the current configuration $\kappa_t$, respectively. The deformation of pre-twist of the tube is described by

\[
R^* = R, \quad \Theta^* = \Theta + \frac{\Theta_0}{L} Z, \quad Z^* = Z,
\]

where $\Theta_0$ is the pre-twisted angle of the tube.

Fig. 5 shows the schematic plots of the morphology of the tube at different tensile loading (cf. the experimental observations in Li and Sun (2002), Sun and Li (2002)). Before and after the phase transition process, the pre-twisted tube is homogeneous and is totally in the austenite or martensite phase (stages 1 and 5). During the phase transition process, after the initiation of the martensite bands (see stage 2 in Fig. 5), the bands will merge under further loading (corresponding to stage 3). When the phase interfaces move close to the two ends, the inclination of the interface can vanish (corresponding to stage 4) and the configuration becomes piecewise homogeneous. If neglecting the inclined effect of the interface, state 3 can also be approximately regarded as a piecewise homogeneous configuration. The tensile stress values at stages 2–4 are almost the same, all of which locate on the stress plateau. In this regard, we analytically model stage 4 to derive the propagation stress of the forward phase transition process. For the unloading process, the configuration of the tube evolves in a reverse manner as that in the loading process. Thus, we also focus on the piecewise homogeneous configuration of the tube during the unloading process (similar to that shown in stage 4 of Fig. 5). That is, the stress at this stage can be regarded as the propagation stress of the reverse phase transition process. Based on these analyses, we only consider homogeneous or piecewise homogeneous configuration of the tube during the whole loading/unloading processes.

The deformation of the pre-twisted tube subjected to a uniaxial tensile force is defined as

\[
r = (1 + W)R^*, \quad \theta = \Theta^*, \quad z = (1 + V)Z*,
\]

where both the radial strain $W$ and axial strain $V$ are some constants independent of $R$ and $Z$. Therefore, the deformation gradient tensors $F^*$ corresponding to the mapping from $\kappa_0$ to $\kappa_0^*$ and $\tilde{F}$ corresponding to the mapping from $\kappa_0^*$ to $\kappa_t$ are of the forms

\[
(F^*) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & M^* \\ 0 & 0 & 1 \end{pmatrix},
\]

Table 1
The shear stress and shear strain data taken from Fig. 6(a) in Sun and Li (2002).
The seven material parameters determined from the experimental data in Table 1.

<table>
<thead>
<tr>
<th>$s_0$</th>
<th>$C_{44}^{(1)}$ (MPa)</th>
<th>A</th>
<th>$D_{11}^*$</th>
<th>$D_{12}^*$</th>
<th>$D_{13}^*$</th>
<th>$D_{55}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0084</td>
<td>18819</td>
<td>0.0194</td>
<td>$6.97 \times 10^{-5}$</td>
<td>$-1.56 \times 10^{-5}$</td>
<td>$4.79 \times 10^{-5}$</td>
<td>$-1.48 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

The expression for \( \mathbf{F} \) can be obtained

\[
(\mathbf{F}) = \begin{pmatrix}
1 + W & 0 & 0 \\
0 & 1 + W & 0 \\
0 & 0 & 1 + V
\end{pmatrix},
\]

where \( M^* := \Theta_0 R/L = \Theta_0 (A + B)/(2L) \) represents the pre-shear strain. The total deformation gradient is then given by \( \mathbf{F} = \mathbf{FF}^* \).

Based on the concept of the multiple natural configurations, we adopt the following form of the deformation gradient tensor \( \mathbf{G} \) corresponding to the mapping from \( \kappa_0 \) to \( \kappa_1 \):

\[
(\mathbf{G}) = \begin{pmatrix}
1 - s_1 & 0 & 0 \\
0 & 1 - s_1 & s_3 \\
0 & 0 & 1 + s_2
\end{pmatrix},
\]

where \( s_1 (i = 1, 2, 3) \) represents the three transformation strains along the radial direction, the axial direction and the shear (circumferential) direction with \( 0 < s_i < 1 \), respectively. Then \( \mathbf{G} \) can be obtained by using (2). By further using (3) and (11), the following infinitesimal strain tensor corresponding to \( \mathbf{F} \) can be obtained

\[
(\mathbf{e}_a) = \begin{pmatrix}
W + a\mathbf{s}_1 \\
0 \\
W + a\mathbf{s}_1
\end{pmatrix} + \begin{pmatrix}
(1 + W)[M^*(a_1 - 1) + a_3] \\
2(a_1 - 1)(1 + a_2) \\
V - a_2
\end{pmatrix}. 
\]

The expressions of \( \sigma^{(i)} := \mathbf{C}^{(i)} : \mathbf{e}_a \) can be derived by using the Voigt notation (similar to that in (18)). Moreover, the elastic energy function can be obtained from (12)

\[
\Phi_l = \left( C_{11}^{(i)} + C_{12}^{(i)} \right) (W + a\mathbf{s}_1)^2 + \frac{2C_{13}^{(i)}(W + a\mathbf{s}_1)(V - a\mathbf{s}_2)}{(1 - a\mathbf{s}_1)(1 + a\mathbf{s}_2)} + \frac{C_{33}^{(i)}(V - a\mathbf{s}_2)^2}{2(1 + a\mathbf{s}_2)^2} + \frac{C_{44}^{(i)}(1 + W)^2}{2(a_1 - 1)^2(1 + a_2)^2}. 
\]

By further using (13), the nominal stress tensor is given by

\[
(\mathbf{S}) = \begin{pmatrix}
S_{Rr} & 0 & 0 \\
0 & S_{r\theta} & S_{r\varphi} \\
0 & S_{2\theta} & S_{2\varphi}
\end{pmatrix},
\]

where the explicit expressions of nonzero stress components can be obtained, for example

\[
S_{2\varphi} = (s_1 \alpha - 1)^2 \left[ (1 - \alpha) \left( \frac{C_{33}^{(1)}(V - s_2\alpha)}{s_2\alpha + 1} - \frac{2C_{13}^{(1)}(W + s_1\alpha)}{s_1\alpha + 1} \right) + \alpha \left( \frac{C_{33}^{(2)}(V - s_2\alpha)}{s_2\alpha + 1} - \frac{2C_{13}^{(2)}(W + s_1\alpha)}{s_1\alpha + 1} \right) \right].
\]

We further consider the following traction-free boundary conditions on the lateral surface of the tube

\[
S_{Rr}|_{R=r_A} = 0, \quad S_{Rr}|_{R=r_B} = 0, \quad S_{2\varphi}|_{R=r_A} = 0, \quad S_{2\varphi}|_{R=r_B} = 0.
\]

It is easy to verify that the last two boundary conditions (41) are automatically satisfied. Suppose the tube is subjected to a tensile force \( \mathcal{F} \) on its end surfaces. Due to the traction-free boundary conditions, the resultant force on each cross section of the tube is also \( \mathcal{F} \). Thus we have

\[
\mathcal{F} = \int_0^B \int_A^{2\pi} S_{2\varphi} RdRd\theta = \pi (B^2 - A^2) S_{2\varphi} \quad \Rightarrow \quad S_{2\varphi}(W, V, \alpha) = \sigma := \frac{\mathcal{F}}{\pi (B^2 - A^2)}.
\]
where $\sigma$ represents the tensile stress on the end surfaces. From (40) and (42), we can obtain the tensile stress–tensile strain relation with other two unknowns $W$ and $\alpha$ to be determined.

In the next section, we use the traction-free boundary conditions (41)1, 2, with the aid of the asymptotic expansion method, to deduce the relation between $W$ and $V$. Then we derive the relation between $\alpha$ and $V$ by using the phase transition criteria (8).

6. One-dimensional asymptotic tensile stress–tensile strain relations

First, we make use of the small strains and adopt the following scalings

$$s_1 = \varepsilon N_1, \ s_2 = \varepsilon N_2, \ s_3 = \varepsilon N_3, \ W = \varepsilon \tilde{W}, \ V = \varepsilon \tilde{V}, \ M^* = \varepsilon \tilde{M}^*,$$

where $\varepsilon$ represents the characteristic axial strain (which is viewed as a small quantity).

By utilizing the new dimensionless variables and solving (41)1, 2 for $W$, we expand the solution in terms of $\varepsilon$ to $O(\varepsilon)$. For simplicity, we assume that the difference between the stiffness matrices of the two strain energy functions is small and can be neglected, i.e. $C_{ij}^{(1)} = C_{ij}^{(2)} = C_{ij}$, then the solution for $W$ is reduced to
By substituting (44) into (40) and using (42), and considering the expansions to $O(\varepsilon^2)$, we obtain the one-dimensional tensile stress–tensile strain relation

$$
\sigma = \epsilon \left( \frac{2C_{13}^2 - (C_{11} + C_{12})C_{33}}{C_{11} + C_{12}} \right) N_2 a - \frac{C_{13} \tilde{V}}{C_{11} + C_{12}} + \epsilon^2 \left( \frac{2C_{13}^2 - (C_{11} + C_{12})C_{33}}{C_{11} + C_{12}} \right) (2N_1 + N_2) (\alpha \tilde{V} - N_2 a \alpha^2).
$$

(45)

In (45), the relation between $\alpha$ and $\tilde{V}$ still needs to be determined. As we have discussed in the previous section, the tensile stress–tensile strain relations for purely austenite and martensite phases can be directly determined by setting $\alpha = 0, 1$ in (45). Next, we take the loading process as an example to show how to obtain the asymptotic expression of $\alpha$ during phase transition by using the phase transition criteria of (8).

The strain energy is $O(\varepsilon^2)$, and it is reasonable to assume that the material parameters $B, \phi_1, \phi_2, k^\pm$ and $Y^\pm$ have the same order. Therefore, we introduce the following scalings

$$
B = \epsilon^2 B, \quad \phi_1 = \epsilon^2 \phi_1, \quad \phi_2 = \epsilon^2 \phi_2, \quad k^\pm = \epsilon^2 k^\pm, \quad Y^\pm = \epsilon^2 Y^\pm.
$$

(46)

Recasting (4) and (38) with the dimensionless variables and by using (7), we obtain the asymptotic expansion of the driving force $D$ up to $O(\varepsilon^4)$, we have

$$
D = \left\{ \alpha \left[ \frac{2B}{C_{11} + C_{12}} - \frac{C_{13} (N_2 a - \tilde{V})}{C_{11} + C_{12}} + \frac{C_{13} \tilde{V}}{C_{11} + C_{12}} \right] \right\} - \frac{C_{44} N_2^2}{2} + \frac{C_{13} C_{44} N_2 M^*}{C_{11} + C_{12}} - \frac{C_{13} C_{44} N_2 M^*}{C_{11} + C_{12}} + \frac{C_{13} C_{44} N_2 M^*}{C_{11} + C_{12}} \right\}
$$

$$
- \frac{2C_{44} N_2 M^* (C_{11} + C_{12}) (2N_1 + N_2) - C_{13} N_2}{C_{11} + C_{12}} + \frac{1}{2} \frac{C_{44} (2N_1 + N_2) M^2}{2 (C_{11} + C_{12})} - \frac{2C_{13} C_{44} N_2 M^*}{C_{11} + C_{12}} \right\}
$$

(47)

By substituting (6), and (47) into (8), we can obtain a quadratic equation for $\alpha$. Solving this equation and expanding the reasonable root to $O(\varepsilon)$, we get the asymptotic expression of $\alpha$ for loading process

$$
\alpha ^+ (\tilde{V}) = L_0^+ + L_1^+ M^* + L_2^+ \tilde{V} + \epsilon \left[ L_3^+ + L_4^+ M^* + L_5^+ M^* \right] \tilde{V} + L_8^+ \tilde{V}^2,
$$

(48)

where the coefficients $L_i^+$ can be expressed in terms of the material parameters (see Appendix). In order to complete the derivation of the asymptotic expression of $\alpha$, the asymptotic expressions of the two critical tensile strains $V_0^+ \tilde{V}$ and $V_1^+ \tilde{V}$ corresponding to the start and end of the phase transition should be derived. These expressions can be achieved by solving equations $\alpha (V) = 0, 1$ for $V$ and choosing the reasonable roots.

The unloading process can be handled similarly. Finally, we obtain the asymptotic expressions of the phase state $\alpha$ for the loading and unloading processes with three regions,

$$
\alpha^\pm = \begin{cases} 0 & \text{if } V < V_0^+, \\ L_0^+ + L_1^+ M^* + L_2^+ \tilde{V} + \epsilon \left[ L_3^+ + L_4^+ M^* + L_5^+ M^* \right] \tilde{V} + L_8^+ \tilde{V}^2 & \text{if } V_0^+ \leq V \leq V_1^+, \\ 1 & \text{if } V > V_1^+. \end{cases}
$$

(49)

where the expressions of $L_i^+$ and the critical tensile strains $V_0^+, V_1^+$ are given in Appendix.

With the asymptotic expressions of $\alpha$, we can derive the one-dimensional tensile stress–tensile strain relations. Substituting (49) into (45), and reusing the original variables, we obtain the following tensile stress–tensile strain relations

$$
\sigma = \begin{cases} 
E_0 \frac{V}{V_0} + Q_0^+ M^* + Q_0^+ M^* & \text{if } 0 \leq V < V_0^+, \\
E_2 (Q_0^+ + Q_0^+ M^* + Q_0^+ M^* + (Q_0^+ + Q_0^+ M^*) (V - V_0^+) + Q_0^+ (V - V_0^+)^2) & \text{if } V_0^+ \leq V \leq V_1^+, \\
E_2 (V - s_2) (1 - 2s_2) & \text{if } V > V_1^+. 
\end{cases}
$$

(50)
where $E_z := [(C_{11} + C_{12})C_{33} - 2C_{13}^2]/(C_{11} + C_{12})$ is the Young’s modulus, and the coefficients $Q_i^z$ ($i = 1, \ldots, 5$) are some material constants. Here and henceforth, we adopt the volume-preserving condition for $G$, i.e. $\det(G) = (1 - s_1)^2(1 + s_2) = 1$.

By considering the leading term, we have

$$E_r = E_\theta, \quad \nu_{\theta r} = \nu_{r\theta}, \quad \nu_{2r} = \nu_{r2}, \quad \frac{\nu_{2r}}{E_z} = \frac{\nu_{r2}}{E_r}$$

which are related with the stiffness by

$$C_{11} = E_r(1 - \nu_{2r}\nu_{r2})Y, \quad C_{12} = E_r(\nu_{\theta r} + \nu_{r2}\nu_{2r})Y,$$

$$C_{13} = E_r(1 + \nu_{\theta r})\nu_{2r}Y = E_r(1 + \nu_{\theta r})\nu_{2r}Y, \quad C_{33} = E_z(1 - \nu_{r\theta}^2)Y,$$

$$C_{44} = \mu_z, \quad Y = \frac{1}{1 - \nu_{r\theta}^2 - 2\nu_{r2}\nu_{2r}(1 + \nu_{\theta r})}.$$ 

For convenience, we adopt the following notations for the material parameters $B, \phi_1, \phi_2, k^z, Y^z, \mu_z$ and $E_z$

$$\omega_1 = \frac{\mu_z}{E_z}, \quad D^z_\phi = \frac{B + \Delta \phi + Y^z}{E_z}, \quad D^z_k = \frac{2B - k^z}{E_z}, \quad D^z_\phi = \frac{B + \Delta \phi - k^z - Y^z}{E_z}.$$ 

where $\Delta \phi := \phi_2 - \phi_1$. Then we can replace the stiffness $C_i$ by the above engineering constants, and the coefficients $Q_i^z$ and $V_i^z$ can be represented in terms of the notations $D^z_\phi, D^z_k, \omega_1, \nu_{2r}, s_3$, and $s_2$ as to be

$$Q^z_0 = \frac{D^z_\phi - D^z_\phi}{s_2} - \frac{D^z_\phi}{s_2^2},$$

$$Q^z_1 = \frac{\left[2D^z_\phi(\nu_{2r} + 1) - s_2\right]s_3\omega_1}{s_2^2},$$

$$Q^z_2 = -\omega_1 \left[s_2^3\omega_1(2\nu_{2r} + 1) + 1\right],$$

$$Q^z_3 = \frac{2s_2D^z_\phi\left(2D^z_k - s_2^3\omega_1(\nu_{2r} + 2)\right)}{\left(-D^z_k + s_2^3\omega_1 + s_2^2\right)^2 + D^z_\phi - s_2^3\omega_1 - s_2^2},$$

$$Q^z_4 = \frac{2s_2s_3\omega_1\left[2\left(s_2^3\omega_1(2\nu_{2r} + 1) + s_2^2\right) - D^z_k(2\nu_{2r} + 2)\right]}{\left(-D^z_k + s_2^3\omega_1 + s_2^2\right)^2},$$

$$Q^z_5 = \frac{s_2^3\left[4s_2^3\omega_1D^z_\phi(2\nu_{2r} + 3) + 6Ds_2^2 + s_2^3\omega_1\left(2s_2^3\omega_1(2\nu_{2r} + 3) + 5s_2^2\right)\right]}{2\left(D^z_\phi - s_2^3\omega_1 - s_2^2\right)^3}.$$ 

$$V^z_0 = Q^z_1 + Q^z_1M^* + Q^z_2M^{*2},$$

$$V^z_1 = \frac{-D^z_\phi + D^z_\phi - M^*s_3\omega_1 + s_2^3\omega_1 + s_2^2}{s_2} - \frac{1}{2s_2^2}\left\{\left[-D^z_k + D^z_\phi + s_2^3\omega_1\right]\left[D^z_\phi - D^z_\phi + s_2^3\omega_1(2\nu_{2r} + 1)\right]\right\}$$

$$+ s_2^3\left[4D^z_\phi - 4D^z_\phi + s_2^3\omega_1\right] - 4M^*s_3\omega_1\left[-(\nu_{2r} + 1)\left(D^z_\phi - D^z_\phi\right) + s_2^3\omega_1(2\nu_{2r} + 1) + s_2^2\right]$$

$$+ 2M^{*2}\omega_1\left[s_2^3\omega_1(2\nu_{2r} + 1) + s_2^2\right].$$

**Remark.** $D^z_\phi$ and $D^z_k$ are material constants when the pre-shear strain $M^* = 0$. For the pre-twisted case, $s_3, D^z_\phi$ and $D^z_k$ depend on material constants and $M^*$. But for convenience, we still refer them as material parameters.
7. Analytical formulas for nucleation stress and propagation stress

Hitherto, the dissipation effects caused by the phase state $\alpha$ are incorporated into the above one-dimensional tensile stress—tensile strain system (50), which is similar to the non-monotonic stress—strain relation discussed in the problem of the Ericksen’s bar (see Ericksen, 1975). In this section, we use the results in Ericksen (1975) to derive the analytical formulas for the nucleation and propagation stresses. Ericksen (1975) studied the one-dimensional equilibrium states of an elastic bar under tension/extension with an up-down-up stress—strain curve, which is associated with a non-convex elastic energy function. It was found that for the displacement-controlled loading process (i.e. the so-called hard device), the homogeneous configurations corresponding to the two increasing branches of the strain–stress curve are stable with respect to infinitesimal disturbances. The equilibrium configuration associated with the decreasing branch is a phase mixture state of the two increasing branches. In this stable configuration, the stress remains constant at a critical stress level throughout the bar and the strain is piecewise constant corresponding to a piecewise homogeneous configuration of the bar. This critical stress (which was called the Maxwell stress) can be obtained by equating the areas between the stress plateau and the strain curve, which is associated with a non-convex elastic energy function. So, in a loading process the stress may climb to the maximum stress if disturbances (always present in reality) are sufficiently small. Then, this maximum stress drops to the critical stress and the deformed configuration becomes piecewise homogeneous.

Our present problem can be viewed as an elastic problem with a non-convex energy function. By using (50), the energy function can be defined as

$$\mathcal{E} := \int_0^l W(V)\,dZ,$$

where

$$W(V) = \begin{cases} 
\frac{1}{2}V^2 & \text{if } 0 \leq V < V_0^\ast, \\
C_1^\ast + \left(Q_0^\ast + Q_3^\ast M^* + Q_5^\ast M^*^2\right)(V - V_0) + \frac{1}{2}(Q_5^\ast + Q_3^\ast M^*)(V - V_0)^2 + \frac{1}{3}Q_5^\ast(V - V_0)^3 & \text{if } V_0^\ast \leq V \leq V_1^\ast, \\
C_2^\ast + \frac{1}{2}(1 - 2s_2)(V - s_2)^2 & \text{if } V > V_1^\ast.
\end{cases}$$

Here, $C_1^\ast$ and $C_2^\ast$ are some constants for the continuity of $W(V)$. By minimizing $\mathcal{E}$, the system (50) can be recovered. Furthermore, for a sufficiently small pre-shear stress, the tensile stress—tensile strain curve is still non-monotonic with three branches like those in Ericksen’s problem, which has been revealed by the experimental results in Sun and Li (2002). Therefore, we can relate our present problem to the problem of Ericksen’s bar and apply the results in Ericksen (1975) to obtain the analytical formulas for the nucleation and propagation stresses of the pre-twisted tube under tension.

For the displacement-controlled loading process, the stress first increases to the martensite nucleation stress $\sigma_{NM}$ (i.e. the local maximum stress at the initiation of the forward transformation), and then it drops to the martensite propagation stress $\sigma_p^M$ (i.e. the critical stress or the Maxwell stress in Ericksen (1975)) and then propagates constantly until the phase transition finishes. On the other hand, for the displacement-controlled unloading process, the stress first decreases to the austenite nucleation stress $\sigma_{NA}\sigma_{NA}$ (i.e. the local minimum at the initiation of the reverse transformation), then jumps to the austenite propagation stress $\sigma_p^A$ and keeps constant until the finish of the phase transition. Both $\sigma_p^A$ can be obtained by equating the areas between their stress levels and the stress—strain curves.

Consequently, we can derive the following nucleation stresses by setting $V = V_0^\ast$ and $V = V_1^\ast$ in (50)

$$\sigma_{NM} = E_s \left[ \frac{D_0^\ast - \omega_1 s_3 M^*}{s_2} - \frac{D_0^2}{s_2} - 2\omega_1 s_3 D_0^2 p_{2zr} + 1\right] M^* + \omega_1 \left(\omega_1 s_3^2 (2p_{2zr} + 1) + s_2^2\right) M^*^2, \quad (57)$$

$$\sigma_{NA} = E_s \left[ \frac{s_2^2 \omega_1 - D_k^- + D_k^- - M^* s_3 \omega_1}{s_2} - \frac{1}{2s_2} \left(\omega_1 s_3^2 (2p_{2zr} + s_2^2) + 5\right) + 2\left(D_k^- - D_0^-\right)^2 - 4s_2^2 \omega_1 (p_{2zr} + 1) \left(D_k^- - D_0^-\right) - 4M^* s_3 \omega_1 \left(\omega_1 s_3^2 (2p_{2zr} + 1) + s_2^2\right)\right]$$

$$+ 4M^* s_3 \omega_1 \left(\omega_1 s_3^2 (2p_{2zr} + 1) + s_2^2\right) - (p_{2zr} + 1) \left(D_k^- - D_0^-\right) + 2M^* \omega_1 \left(\omega_1 s_3^2 (2p_{2zr} + 1) + s_2^2\right), \quad (58)$$

where $\omega_1$ and $\omega_2$ are some constants for the continuity of $W(V)$. By minimizing $\mathcal{E}$, the system (50) can be recovered.
With the above-described equal area rule and through some calculations, we get the following analytical formulas for \( \sigma_{\beta}^e \):

\[
\sigma_{\beta}^e = E_z\left\{ \frac{s_2^2}{2s_2^2} \omega_1 \left( -\frac{D_k^-}{s_2^2} + 2D_k^+ - 2M^s s_3^2 \omega_1 \right) - \frac{1}{12s_2^2} \omega_1^2 \left( -D_k^+ \left( 4s_2^2 + 3 \right) + 6D_k^- (s_2^2 + 1) + 3 \left( D_k^+ - 2D_k^- \right)^2 \right) + \omega_1 s_2^2 \left( 8s_2^2 + 3 \right) + 10s_2^2 \right\} - \frac{M^s \left[ \omega_1 s_3 (s_2^2 + 1) \left( D_k^+ - 2D_k^- \right) - \omega_1 s_2^2 (2s_2^2 + 1) + 2s_2^2 \right]}{s_2^2}
\]

(59)

8. Determination of material response

In the previous sections, we have obtained the analytical formulas for the tensile stress–tensile strain relations, the nucleation and propagation stresses in terms of the material parameters \( \mu_2, v_2, E_z, s_2, s_3, D_k^+, D_k^- \) (noting that \( \omega_1 = \mu_2/E_z \)) and the pre-shear strain \( M^s \). Although the pre-twisted tension tests on SMA tubes were conducted by Sun and Li (2002), the available data on the propagation stress are not enough. So in order to determine the material response through the propagation stress with different levels of pre-twist, we conduct a series of tension tests with different fixed pre-twists on the polycrystalline thin-walled NiTi tube, with a focus on the stress–strain response.

8.1. Pre-twisted tension experiments on NiTi tube

The material used in the experiment was a commercial polycrystalline NiTi tube with composition of 55.9wt% Ni (Nitinol Devices & Components, USA). The outer and inner diameters of the tube were 1.78 and 1.47 mm, respectively. The grain size was in the range of 50–100 nm. Using differential scanning calorimeter (DSC), the measured austenite finish temperature \( (A_f) \) of the tube was 1°C. So the material is in the austenite state at room temperature (23°C) and will exhibit superelastic behavior under stress. The sample was first chemically etched by hydrofluoric acid into a dog-bone shape and then mechanically polished by fine grained sand papers to reach a final surface roughness of 0.15 mm. The dimensions of the sample used in the experiment of this paper are: the outer diameter of 1.73 mm and the gauge length of 40 mm.

The testing machine as shown in Fig. 6 is developed for tension test with fixed pre-twist on such specimen. There are such four subsystems in the machines as motion control, image acquisition, force/torque measurement, and software. The features of these subsystems are as follows:

- Two servo motors with 17-bit high resolution encoder (Mitsubishi HF-KE43JW1-S100, Japan) are used as the source of motive power. One causes twist of the specimen and the other, together with linear motion parts, causes elongation of the specimen. The corresponding resolution of the imposed nominal tensile and shear strain is about 0.8 με.
- Three cameras with the capacity of \( 1920 \times 1080 @ 25 \)fps high-definition video recording and 18-million-pixel high-resolution photographing (Canon EOS-550D, Japan) are employed. The specimen is sprayed with tiny paint speckles carrying displacement information and high-efficiency algorithms of digital image processing techniques such as pattern matching and target tracking are proposed and lead to non-contact, direct, tension–torsion-morphology synchronized, and full-field strain measurement. The measurement resolution can be as high as 0.3 με.
- Force and torque are measured by a combined transducer (TML customized SLP-500NS, Japan) in which a single elastic component and strain gauge rosettes are used.
- The software is developed in the environment of Microsoft Visual C++ 2005. Its main functions are manual motor control, automatic motor control, real-time display of force and torque data, and data access.

We set 11 levels of pre-twist, so there are totally 11 different tests in one series. In every test, the maximum tension is high enough to induce phase transformation even without the pre-twist. The time consumed in each test is in the range of 9–11 min for the balance between the latent heat (whose influence decreases with the testing time) and the mass image data (which prefer short-time testing to reduce computational load). Four tubing specimens are tested to guarantee the repeatability of experimental results. The nominal normal and shear strains are measured by video and obtained by tracking the location of characteristic speckles on the two ends of the specimen. The nominal normal and shear stresses are calculated from the measured force and torque and the specimen geometry. The measured tensile stress–tensile strain curves with 11 levels of pre-twist are shown in Fig. 7. The results of Ericksen (1975) suggest that the instability of the material is associated with the observed drop in tensile stress near the onset of the forward phase transformation. From Fig. 7, one can see that when the pre-shear strain \( M^s \) is at a relatively low level, the tensile stress–tensile strain curve is unstable with a stress plateau (test no. 1–5). With the increase of \( M^s \), the response gradually becomes monotonically hardened (test no. 6–11).
8.2. Determining material response through the pre-twisted tension tests

Now we use the experimental data to determine the seven material parameters $\mu_2, \nu_2, E_z, S_2, S_3, D_1^+, D_2^+$. As the pre-twisting effect may influence the microstructure of the tube, it is necessary to stipulate the relations between the seven material parameters and the shear strain $M^*$. 

![Figure 6. Principal parts of the testing machine for tension/torsion test.](image)

![Figure 7. The measured tensile stress–tensile strain curves of tension tests with different levels of fixed pre-twist. $M^*$ and $\tau^*$ are the pre-shear strain and stress respectively.](image)
First, by definitions \( \mu_{2r} \), \( E_S \), and \( \mu_2 \) are independent of \( M^* \). It is found that \( \mu_{2r} = 0.33 \). From the curve of test no. 1 in Fig. 7, we read the data in the axial of stress at the strain of 0.65% and get the Young’s modulus \( E_S = 28429 \text{MPa} \). From the pre-shear strain and pre-shear stress of test no. 2 in Fig. 7, we get \( \mu_2 = 16184 \text{MPa} \). As indicated in the experiment, the transformation strain \( s_2 \) of the tube at different pre-twisted levels under tension is relatively small. Therefore, we assume that \( s_2 \) is independent of \( M^* \). We set \( s_2 = 0.047 \) which is recorded in the pure tension test.

Thus, there only remains three material parameters \( D^F_\phi, D^F_k \) and \( s_3 \) to be determined. In practice, there is no difference of the tensile stress for the two opposite directions of the pre-twist on the ends of the tube (i.e. \( M^* > 0 \) or \( M^* < 0 \)). So \( \sigma \) should be an even function of \( M^* \). By the definition, \( s_3 \) should be an odd function of \( M^* \). From (50) and (54), it can be found that \( D^F_\phi \) and \( D^F_k \) should be even functions such that \( \sigma \) is an even function of \( M^* \). Further, we treat them as continuous and differentiable functions. Then, from their Taylor expansions in small \( M^* \), \( D^F_\phi, D^F_k, s_3 \) are approximately taken as the following functions of the pre-shear strain \( M^* \):

\[
\begin{align*}
D^F_\phi(M^*) &= b_1 + k_1 M^*^2, \\
D^F_k(M^*) &= b_2 + k_2 M^*^2, \\
s_3(M^*) &= k_3 M^*,
\end{align*}
\]

(60)

where \( b_1 := D^F_\phi(0), b_2 := D^F_k(0), \) and \( k_1 - k_3 \) are five constants to be determined. We read the five values of the propagation stress of test no. 1–5 in Fig. 7 (see Table 3). Then using the analytical formula of the propagation stress (59), the five unknowns can be easily determined by some simple numerical methods as all equations are entirely algebraic. Here, we make use of the Nelder-Mead method (see Nelder and Mead, 1965) and finally obtain the following values for the five material constants

\[
\begin{align*}
b_1 &= 3.66 \times 10^{-4},
b_2 &= 6.775 \times 10^{-5},
k_1 &= 0.37,
k_2 &= 1.43,
k_3 &= 2.68.
\end{align*}
\]

(61)

Thus the tensile stress–tensile strain relation for loading process is determined as follows:

\[
\sigma = \begin{cases} 
57000 V & \text{if } 0 \leq V < 0.00773, \\
\sigma^{pt}(V; M^*) & \text{if } 0.00773 \leq V \leq 0.0539, \\
51642(V - 0.047) & \text{if } V > 0.0539,
\end{cases}
\]

(62)

where

\[
\sigma^{pt}(V; M^*) = \frac{1}{(0.00351 + M^*^2)^3} \left[ (1.97 - 7.564 V - 0.767 V^2) \times 10^{-5} + M^*^2 ( - 0.0116 + 0.886 V - 6.08 V^2) \\
+ M^*^4 ( - 9.811 - 392.692 V - 1076.45 V^2) + M^*^6 ( - 1656.12 + 51973.4 V) \right].
\]

(63)

By setting \( M^* = 0 \) in (63), the softening tensile stress–tensile strain curve of the tube under pure tension is captured (see Fig. 8). The distinction of the nucleation stress and the propagation stress can be found in Fig. 8.

To capture such intrinsic material response of SMAs over the stress plateau, Hallai and Kyriakides (2013) conducted uniaxial tensile tests on a laminate (consisting of two stainless steel strips as the face-strips and one NiTi layer as the core) and the stainless steel, respectively. Under proper design, the instability of the core in such laminates was found to be suppressed by the hardening effect of the face-strips. This was structurally reflected in the recorded monotonic stress–strain curves of the laminate. Subsequently, they extracted the material response of NiTi from the response of the laminate and the face-strips, which was found to have an up-down-up fashion. Moreover, Song et al. (2013) used the same constitutive theory as the present work to study phase transformations of a NiTi wire under uniaxial tension based on a three-dimensional setting. Also using the asymptotic expansion method and the equilibrium theory in Ericksen (1975), they deduced the analytical formulas of the nucleation and propagation stresses. Subsequently, they used the measurements of both the nucleation and the propagation stresses to capture the up-down-up nominal stress–strain curve. However, the value of nucleation stress is generally difficult to be measured in the experiments. Here we provide another approach to find the true material response of SMAs.

Besides that, in order to validate the pre-shear effects, we use the determined material constants as well as (57) and (59) to get the nucleation and propagation stresses and the stress drop \( (\sigma_d = \sigma_{NM} - \sigma^p_\phi) \) as follows:

Table 3

The stress data from Fig. 7.

<table>
<thead>
<tr>
<th>( M^*(%) )</th>
<th>0</th>
<th>0.2762</th>
<th>0.5336</th>
<th>0.7996</th>
<th>1.0775</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma^p_\phi ) (MPa)</td>
<td>400</td>
<td>395.5</td>
<td>392.8</td>
<td>390.7</td>
<td>388.3</td>
</tr>
</tbody>
</table>
provides another method for better controllability: If the tube is pre-loaded by a shear strain (higher than graded SMAs to avoid instability such that better controllability in actuation applications is achieved. The present work formations to homogeneous deformations in the experiments. Note that Shariat et al. (2012, 2013) created functionally no. 6). On the other hand, this determined critical shear strain can conversely predict the turning point from localized de-

Table 4. Using the analytical formulas of the nucleation and propagation stresses (57) and (59) and the relation $M^*$ nucleation stress value. On the other hand, on the stress plateau, the con

As can be seen in Fig. 9, the propagation stresses at only four pre-shear stresses from 0

Fig. 8. The determined up-down-up material stress–strain relation.

$$\sigma_{NM} = 440.472 - 4.782 \times 10^5 M^{-2} - 9.1 \times 10^6 M^{-4},$$

$$\sigma^*_p = 400 - 1.307 \times 10^5 M^{-2} - 1.575 \times 10^5 M^{-4},$$

$$\sigma_d = 40.472 - 3.475 \times 10^5 M^{-2} - 7.525 \times 10^6 M^{-4}. \quad (64)$$

The nucleation stress value by the above formula is considerably larger than the measured peak stress given in Fig. 7. This is due to the instability near the onset of forward transformation; In an experiment the peak stress never reaches the nucleation stress value. On the other hand, on the stress plateau, the configuration is stable. So the propagation stress approximates the plateau stress. The above relations imply that both the nucleation and the propagation stresses decrease with $M^*$ increasing, but the descending speed of $\sigma_p$ is smaller than that of $\sigma_{NM}$. It leads to the result that the stress drop decreases with $M^*$ increasing, which is consistent with the above function $\sigma_d(M^*)$. When the pre-shear strain $M^*$ increases to a critical value $M^*_c$, the stress drop vanishes. At this moment, the deformations turn to be stable. Setting $\sigma_d = 0$ in (64) and solving it for $M^*_c$, we get $M^*_c = 0.0108$, which is consistent with that observed in the experiments (the measured stress–strain curve becomes monotonically hardened at a pre-twist level between in test no. 5 and $M^* = 0.010775 M^* = 0.013162$ in test no. 6). On the other hand, this determined critical shear strain can conversely predict the turning point from localized deformations to homogeneous deformations in the experiments. Note that Shariat et al. (2012, 2013) created functionally graded SMAs to avoid instability such that better controllability in actuation applications is achieved. The present work provides another method for better controllability: If the tube is pre-loaded by a shear strain (higher than $M^*_c$), then the response of the pre-twisted tube under tension is stable.

8.3. Determining material response through previous experimental data

Note that Sun and Li (2002) also conducted combined tension–torsion tests on SMAs, in which the specimen is a Ti-56wt% Ni tube. We shall use their experimental data to determine the material response as another validation of the analytical results and the methodology of the present work. Fig. 9 is Fig. 7 of Sun and Li (2002), which shows the measured tensile stress–tensile strain curves of the pre-twisted tube under tension with different pre-shear stresses. The specimen and the test temperature were the same as those in their pure torsion test. Thus from their pure tension test, we read $s_2 = 0.047$. From Fig. 9, we read the data in the axial of stress at the strain of 0.008 and get the Young’s modulus $E_z = 28429 MPa$. From the measured shear modulus of the tube under pure torsion in Table 2, we get $\mu_z = 18819 MPa$. $v_p\mu_p = 18819 MPa v_p$ is also 0.33. As can be seen in Fig. 9, the propagation stresses at only four pre-shear stresses from 0MPa to 134MPa can be clearly read out, while we have five unknowns $b_1 - b_2$ and $k_1 - k_3$ (see Equation (60)). Considering that the deformations become much more stable at certain pre-shear stress as indicated in Sun and Li (2002), we further choose one nucleation stress which is at the pre-shear stress of 184MPa. This chosen measurement of the nucleation stress should be more accurate than the one under pure tension. These measurements of the one nucleation stress value and the other four propagation stress values are listed in Table 4. Using the analytical formulas of the nucleation and propagation stresses (57) and (59) and the relation $M^* = \tau^*/\mu_z$ the five unknowns can also be determined by using the Nelder-Mead method (see Nelder and Mead, 1965) as follows:

$$b_1 = 8.071 \times 10^{-4}, \quad b_2 = 8.018 \times 10^{-5}, \quad k_1 = 0.593, \quad k_2 = 2.385, \quad k_3 = 2.029. \quad (65)$$
Thus the tensile stress–tensile strain relation for loading process is determined as follows:

\[
\sigma = \begin{cases} 
28429V & \text{if } 0 \leq V < 0.0169, \\
\sigma^p(V; M^*) & \text{if } 0.0169 \leq V \leq 0.0637, \\
25756.7(V - 0.047) & \text{if } V > 0.0637, 
\end{cases}
\]

where

\[
\sigma^p(V; M^*) = \frac{1}{(0.00213 + 0.34M^*)^3} \left[ \left( -1.211V^2 - 9.552V + 4.79 \right) \times 10^{-6} - M^* \left( 0.926V^2 - 0.098V + 0.00365 \right) \\
- M^4 \left( 60.232V^2 - 2.045V + 0.769 \right) - M^6(202.069 - 411.162V) \right].
\]

(67)

By setting \( M^* = 0 \) in (67), the softening tensile stress–tensile strain curve of the tube under pure tension is captured (see Fig. 10).

Similarly, we obtain the nucleation and propagation stresses and the stress drop \( \sigma_d = \sigma_{NM} - \sigma_p^+ \) as follows:

\[
\sigma_{NM} = 479.809 - 4.47759 \times 10^5 M^* - 1.57994 \times 10^7 M^*^4, \\
\sigma_p^+ = 456.372 - 3.46376 \times 10^5 M^* - 4.89 \times 10^6 M^*^4, \\
\sigma_d = 23.437 - 1.01383 \times 10^5 M^* - 1.09094 \times 10^7 M^*^4.
\]

(68)

In Fig. 11, the nucleation-stress, the propagation-stress and the stress-drop curves are plotted as functions of \( M^* \), together with the experimental data on the peak stress, the plateau stress, and the corresponding drop in stress in Fig. 9. As expected, the theoretical value for the nucleation stress is greater than the measured peak stress (due to instability, in an experiment the peak stress never reaches the nucleation stress value). Next, by setting \( \sigma_d = 0 \), we get \( M_c^* = 0.015 \) and the corresponding pre-shear stress \( \tau^* = 282.3 \text{MPa} \), which is consistent with that observed in the experiments of Sun and Li (2002). In fact, from Fig. 9, it can be seen that the deformation of the pre-twisted tube becomes homogeneous at a pre-twisted stress level between 278MPa and 287MPa. As a repeated validation of the analytical results and the methodology of the present work, the

<table>
<thead>
<tr>
<th>( \tau ) (MPa)</th>
<th>0</th>
<th>43</th>
<th>90</th>
<th>134</th>
<th>184</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_p ) (MPa)</td>
<td>456.4</td>
<td>451.7</td>
<td>445.6</td>
<td>440.4</td>
<td>_</td>
</tr>
<tr>
<td>( \sigma_d ) (MPa)</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>436.9</td>
</tr>
</tbody>
</table>

Table 4
The stress data from Fig. 7 in Sun and Li (2002).
above determined material response and the critical shear strain for the experiments in Sun and Li (2002) again implies that the present work can be applied to capture the up-down-up material response of SMA tubes. And the results are also useful for avoiding instability in actuation applications by using SMA tubes.

9. Conclusion

In this paper, we mainly focused on the mechanical behavior of an SMA thin-walled tube under torsion and pre-twisted tension. A constitutive model with specific forms of Helmholtz free energy and mechanical dissipation rate was adopted. To consider the anisotropic effect, SMA was treated as a transversely isotropic material. The theory of linear elasticity was used to describe the elastic responses of the austenite and martensite phases. The homogeneous deformation of the SMA thin-walled tube under pure torsion was first studied and the corresponding one-dimensional analytical shear stress—shear strain relation was derived. By using the relevant experimental data, the monotonic hardening shear stress—shear strain response was quantitatively predicted. Furthermore, by considering the homogeneous and piecewise homogeneous deformations of the pre-twisted tube under tension, we derived one-dimensional analytical tensile stress—tensile strain relations in terms of the pre-shear strain and some material constants. By further relating this problem to the equilibrium theory of Ericksen’s bar, the analytical formulas for the nucleation and propagation stresses, as well as the corresponding stress drop were obtained. Tension tests with different pre-twists on SMA thin-walled tube were conducted. By using the measured values and the analytical formula for the propagation stress, the relevant material parameters were determined. Then, the intrinsic material response of the tube under pure tension was successfully captured with an up-down-up fashion. Moreover, the relations between the nucleation stress as well as the propagation stress and the pre-shear strain were revealed. We also determined
the critical value of the pre-shear strain at which the structure response becomes stable. Above this pre-shear strain level, the corresponding tensile stress—tensile strain curves will become monotonically increasing. In addition, the previous experimental data were also used to determine the material response, which repeatedly validates the analytical results and the methodology of the present work.

To sum up, the present work provides an approach to capture the up-down-up material response and suggests a plausible way for a better controllability in actuation applications by using SMA tubes. Finally, we remark that the constitutive model adopted in the present work does not consider the reorientation of martensite variants, due to its simplicity and convenience for deriving the analytical results. In the multi-axial state of stress and non-proportional loading path, reorientation effects can arise. In literature, constitutive models that take into account the effect of martensite reorientation to describe the (pseudoelastic) behavior of SMAs have been proposed (see, Pan et al., 2007; Panico and Brinson, 2007; Arghavani et al., 2010b; Chemisky et al., 2011; Zaki, 2012). With suitable modifications, the present methodology may be applicable to these models, which we leave for a future investigation.

Acknowledgments

The work described in this paper was supported by a GRF grant (Project No.: CityU 11303015) from the Research Grants Council of Hong Kong SAR, China and a grant from the National Nature Science Foundation of China (Project No.: 11572272).

Appendix. Constants in the derivation

The material constants in (29) are given by

\[
\begin{align*}
B_0^r &= \frac{s_0^2}{2C_0^2} (2\Delta D_k^r - D_k^r) + 2D_k^r, \\
B_1^r &= -\Delta s_0 D_k^r, \\
B_2^r &= \frac{s_0^2}{2C_0^2} (8\Delta D_k^r - 4D_k^r + s_0^2) + 4D_k^r, \\
B_3^r &= -2\Delta s_0 (s_0^2 + 2D_k^r), \\
B_4^r &= \Lambda^2 s_0^2.
\end{align*}
\]  

(A.1)

The constants \( L^e \) and \( \tilde{V}^e \) in (49) are given by

\[
\begin{align*}
L_0^e &= \frac{(C_{11} + C_{12})(\tilde{B} + \tilde{Y} - \tilde{\phi}_1 + \tilde{\phi}_2)}{(C_{11} + C_{12})(2\tilde{B} - \tilde{K} - C_{33}N_2^2 - C_{44}N_3^2) + 2C_{13}^2N_2^2}, \\
L_0^e &= \frac{(C_{11} + C_{12})(\tilde{B} - \tilde{\phi}_1 + \tilde{\phi}_2 - \tilde{K} - \tilde{Y})}{(C_{11} + C_{12})(2\tilde{B} - \tilde{K} - C_{33}N_2^2 - C_{44}N_3^2) + 2C_{13}^2N_2^2}, \\
L_1^e &= \frac{(C_{11} + C_{12})C_{44}N_3}{(C_{11} + C_{12})(2\tilde{B} - \tilde{K} - C_{33}N_2^2 - C_{44}N_3^2) + 2C_{13}^2N_2^2}, \\
L_2^e &= \frac{2C_{13}^2 - (C_{11} + C_{12})C_{33}}{(C_{11} + C_{12})(2\tilde{B} - \tilde{K} - C_{33}N_2^2 - C_{44}N_3^2) + 2C_{13}^2N_2^2}, \\
L_3^e &= \frac{(C_{11} + C_{12})^2(\tilde{B} + \tilde{Y} - \tilde{\phi}_1 + \tilde{\phi}_2)^2}{2\left[(C_{11} + C_{12})(2\tilde{B} - \tilde{K} - C_{33}N_2^2 - C_{44}N_3^2) + 2C_{13}^2N_2^2\right]^3} \times \left[C_{44}N_3^2(4(C_{11} + C_{12})N_1 + 3(C_{11} + C_{12}) - 4C_{13}N_2) \right. \\
& \left. - 3(2C_{13}^2 - (C_{11} + C_{12})C_{33})N_2(2N_1 + N_2)\right], \\
L_3^e &= \frac{(C_{11} + C_{12})^2(\tilde{B} - \tilde{\phi}_1 + \tilde{\phi}_2 - \tilde{K} - \tilde{Y} - \tilde{\phi}_1 + \tilde{\phi}_2)^2}{2\left[(C_{11} + C_{12})(2\tilde{B} - \tilde{K} - C_{33}N_2^2 - C_{44}N_3^2) + 2C_{13}^2N_2^2\right]^3} \times \left[C_{44}N_3^2(4(C_{11} + C_{12})N_1 + 3(C_{11} + C_{12}) - 4C_{13}N_2) \right. \\
& \left. - 3(2C_{13}^2 - (C_{11} + C_{12})C_{33})N_2(2N_1 + N_2)\right].
\end{align*}
\]  

(A.2) (A.3) (A.4) (A.5) (A.6) (A.7)
\[ L_4^z = \frac{(C_{11} + C_{12})C_{44}N_3 (B + \tilde{Y}^\pm - \tilde{\phi}_1 + \tilde{\phi}_2)}{[(C_{11} + C_{12}) (2B - \tilde{k}^z - C_{33}N_2^2 - C_{44}N_2^3) + 2C_{13}^2N_2^2]^3} \times \left\{ \begin{array}{l} 4(C_{11} + C_{12})B(2(C_{11} + C_{12})N_1 + (C_{11} + C_{12} - C_{13})N_2) \\ + C_{13}^2 (2(C_{11} + C_{12})N_2^2 + C_{44}N_2^3) + 2C_{13}N_2^2 \\ + 2C_{13}(2(2N_1 + N_2)\tilde{k}^+ + C_{33}(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \end{array} \right\}, \] (A.8)

\[ L_5^z = \frac{(C_{11} + C_{12})C_{44}}{2 [(C_{11} + C_{12}) (2B - \tilde{k}^z - C_{33}N_2^2 - C_{44}N_2^3) + 2C_{13}N_2^2]^3} \times \left\{ \begin{array}{l} -4(C_{11} + C_{12})B((C_{11} + C_{12})(2N_1 + N_2) \\ - C_{13}C_{44}N_2^3) - 2(C_{11} + C_{12})C_{23}^2N_2 (2(2N_1 + N_2)N_2^2 + C_{33}(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 2(C_{11} + C_{12})C_{33}C_{44}N_2^3 (\tilde{k}^+ + C_{33}N_2^2) + (C_{11} + C_{12})^2 C_{44}N_2^3 (2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 2(C_{11} + C_{12})C_{44}N_2^3 (2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 4C_{44}N_2^3 (2(2N_1 + N_2)N_2^2 + C_{44}C_{44}N_2^3 N_2)^3, \end{array} \right\}, \] (A.9)

\[ L_6^z = \frac{(C_{11} + C_{12}) (B + \tilde{Y}^- - \tilde{\phi}_1 + \tilde{\phi}_2)}{[(C_{11} + C_{12}) (2B - \tilde{k}^z - C_{33}N_2^2 - C_{44}N_2^3) + 2C_{13}N_2^2]^3} \times \left\{ \begin{array}{l} 4(C_{11} + C_{12})B((C_{11} + C_{12})C_{33} - 2C_{13}N_2^2) + 2(N_1 + N_2) \\ - C_{13}C_{44}N_2^3) - 2(C_{11} + C_{12})C_{23}^2N_2 (2(2N_1 + N_2)N_2^2 + C_{33}(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 2(C_{11} + C_{12})C_{33}C_{44}N_2^3 (\tilde{k}^+ + C_{33}N_2^2) + (C_{11} + C_{12})^2 C_{44}N_2^3 (2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 2(C_{11} + C_{12})C_{44}C_{44}N_2^3 (\tilde{k}^+ - C_{33}N_2^2 + C_{44}N_2^3 N_2) + 4C_{44}N_2^3 (2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2)^3, \end{array} \right\}, \] (A.10)

\[ L_6^y = \frac{(C_{11} + C_{12}) (B - \tilde{\phi}_1 + \tilde{\phi}_2 - \tilde{k}^z - \tilde{Y}^-)}{[(C_{11} + C_{12}) (2B - \tilde{k}^z - C_{33}N_2^2 - C_{44}N_2^3) + 2C_{13}N_2^2]^3} \times \left\{ \begin{array}{l} 4(C_{11} + C_{12})B((C_{11} + C_{12})C_{33} - 2C_{13}N_2^2) + 2(N_1 + N_2) \\ - C_{13}C_{44}N_2^3) - 2(C_{11} + C_{12})C_{23}^2N_2 (2(2N_1 + N_2)N_2^2 + C_{33}(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 2(C_{11} + C_{12})C_{44}N_2^3 N_2^2 (\tilde{k}^+ - C_{33}N_2^2 + C_{44}N_2^3 N_2) \\ + C_{44}N_2^3 (2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 4C_{44}N_2^3 (2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2)^3, \end{array} \right\}, \] (A.11)

\[ L_7^z = \frac{(C_{11} + C_{12})C_{44}N_3}{[(C_{11} + C_{12}) (2B - \tilde{k}^z - C_{33}N_2^2 - C_{44}N_2^3) + 2C_{13}N_2^2]^3} \times \left\{ \begin{array}{l} 4B (2(C_{11} + C_{12})C_{13}N_2^2) - (C_{11} + C_{12})C_{13}C_{44}N_2^3 \\ + 2C_{44}N_2^3) - 2(C_{11} + C_{12})C_{13}N_2 (4(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) + 2C_{13}(2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2) \\ + 4(C_{11} + C_{12})C_{13} (\tilde{k}^+ + C_{44}N_2^3 N_2) + 4C_{44}N_2^3 (2(2N_1 + N_2)N_2^2 + C_{44}N_2^3 N_2)^3, \end{array} \right\}. \] (A.12)
\[ L_8^+ = \frac{(C_{11} + C_{12}) \left( 2C_{13}^2 - (C_{11} + C_{12})C_{33} \right)}{2 \left[ (C_{11} + C_{12}) \left( 2B - \tilde{k}^+ - C_{33}N_2^2 - C_{44}N_3^2 \right) + 2C_{13}^2 N_2^2 \right]} \times \left[ 4B \left( C_{11} + C_{12} \right) \left( 2N_1 + N_2 \right) \left( - \tilde{k}^+ + C_{33}N_2^2 - C_{44}N_3^2 \right) \right. \\
- \left. 2C_{13}^2 \left( 2N_1 + N_2 \right) N_2^2 - 2C_{13}C_{44}N_3^2 N_2 \right) + 4(C_{11} + C_{12}) \left( 2N_1 + N_2 \right) B^2 + 4C_{13}C_{44}N_2N_3 \left( \tilde{k}^+ + C_{44}N_3^2 \right) \right. \\
+ \left. 2C_{13}^2N_2^2 \left( 2(2N_1 + N_2) \tilde{k}^+ - C_{44}N_3^2 N_2 \right) + (C_{11} + C_{12}) \left( C_{33}N_2^2 \left( C_{44}N_2N_3^2 - 2(2N_1 + N_2) \tilde{k}^+ \right) + (2N_1 + N_2) \left( \tilde{k}^+ + C_{44}N_3^2 \right)^2 \right) \right], \\
(A.13) \]

\[ L_8^- = \frac{(C_{11} + C_{12}) \left( 2C_{13}^2 - (C_{11} + C_{12})C_{33} \right)}{2 \left[ (C_{11} + C_{12}) \left( 2B - \phi_1 + \phi_2 - \tilde{k}^- - \tilde{Y}^- \right) + 2C_{13}^2 N_2^2 \right]} \times \left[ 4B \left( C_{11} + C_{12} \right) \left( 2N_1 + N_2 \right) \left( - \tilde{k}^- + C_{33}N_2^2 - C_{44}N_3^2 \right) \right. \\
- \left. 2C_{13}^2 \left( 2N_1 + N_2 \right) N_2^2 - 2C_{13}C_{44}N_3^2 N_2 \right) + 4(C_{11} + C_{12}) \left( 2N_1 + N_2 \right) B^2 + 4C_{13}C_{44}N_2N_3 \left( \tilde{k}^- + C_{44}N_3^2 \right) \right. \\
+ \left. 2C_{13}^2N_2^2 \left( 2(2N_1 + N_2) \tilde{k}^- - C_{44}N_3^2 N_2 \right) + (C_{11} + C_{12}) \left( C_{33}N_2^2 \left( C_{44}N_2N_3^2 - 2(2N_1 + N_2) \tilde{k}^- \right) + (2N_1 + N_2) \left( \tilde{k}^- + C_{44}N_3^2 \right)^2 \right) \right], \\
(A.14) \]

\[ \tilde{V}_0^+ = \frac{(C_{11} + C_{12})}{\left( C_{11} + C_{12} \right) \left( C_{33} - 2C_{13}^2 \right) N_2} \left[ \left( B - \tilde{\phi}_1 + \tilde{\phi}_2 + \tilde{Y}^+ \right) - C_{44}N_3 \tilde{M}^* \right] - \frac{\varepsilon}{2N^2} \left( C_{44} \tilde{M}^* \left( \left( C_{11} + C_{12} \right) \left( C_{33} - 2C_{13}^2 \right) \right) \right) \right. \\
\left( 2N_1 + N_2 \right) N_2^2 - C_{44}N_3^2 \left( 4C_{13}N_2 + (C_{11} + C_{12}) \left( 2N_1 + N_2 \right) \right) - 2C_{44}N_3 \tilde{M}^* \left( \left( C_{11} + C_{12} \right) \left( 2N_1 + N_2 \right) + 2C_{13}N_2 \right) \left( B + \tilde{Y}^+ - \tilde{\phi}_1 + \tilde{\phi}_2 \right) + (C_{11} + C_{12}) \left( 2N_1 + N_2 \right) \left( B + \tilde{Y}^+ - \tilde{\phi}_1 + \tilde{\phi}_2 \right)^2 \right]. \\
(A.15) \]

\[ \tilde{V}_0^- = \frac{(C_{11} + C_{12})}{\left( C_{11} + C_{12} \right) \left( C_{33} - 2C_{13}^2 \right) N_2} \left[ \left( B - \tilde{\phi}_1 + \tilde{\phi}_2 - \tilde{k}^- - \tilde{Y}^- \right) - C_{44}N_3 \tilde{M}^* \right] - \frac{\varepsilon}{2N^2} \left( C_{44} \tilde{M}^* \left( \left( C_{11} + C_{12} \right) \left( C_{33} - 2C_{13}^2 \right) \right) \right) \right. \\
\left( 2N_1 + N_2 \right) N_2^2 - C_{44}N_3^2 \left( 4C_{13}N_2 + (C_{11} + C_{12}) \left( 2N_1 + N_2 \right) \right) - 2C_{44}N_3 \tilde{M}^* \left( \left( C_{11} + C_{12} \right) \left( 2N_1 + N_2 \right) + 2C_{13}N_2 \right) \left( B - \tilde{\phi}_1 + \tilde{\phi}_2 - \tilde{k}^- - \tilde{Y}^- \right)^2 \right]. \\
(A.16) \]

\[ \tilde{V}_1^+ = \frac{(C_{11} + C_{12})}{\left( 2C_{13}^2 - (C_{11} + C_{12})C_{33} \right) N_2} \left[ \left( B - \tilde{k}^- + \tilde{Y}^+ + \tilde{\phi}_1 - \tilde{\phi}_2 - C_{33}N_2^2 - C_{44}N_3^2 \right) + 2C_{13}N_2^2 + C_{44}N_3 \tilde{M}^* \right] \\
+ \frac{(C_{11} + C_{12})\varepsilon}{\left( C_{11} + C_{12} \right) \left( C_{33} - 2C_{13}^2 \right) \left( 2N_2^2 \right)^2} \times \left( 2C_{44}N_3 \tilde{M}^* \left( \left( - \tilde{B} + \tilde{k}^- + \tilde{Y}^+ - \tilde{\phi}_1 + \tilde{\phi}_2 \right) \left( 2C_{13}N_2 + (C_{11} + C_{12}) \left( 2N_1 + N_2 \right) \right) \right) \right. \\
+ \left. \left. \left( 4C_{13}N_2 + (C_{11} + C_{12}) \left( 2N_1 + N_2 \right) \right) \left( C_{44}N_3^2 \right) + \left( C_{11} + C_{12} \right) \left( C_{33} - 2C_{13}^2 \right) \left( 2N_1 + N_2 \right) N_2^2 \right) - \left( \left( - \tilde{B} + \tilde{k}^- + \tilde{Y}^+ - \tilde{\phi}_1 + \tilde{\phi}_2 \right) \left( 2C_{13}N_2 + (C_{11} + C_{12}) \left( 2N_1 + N_2 \right) \right) \left( C_{44}N_3^2 \right) + \left( C_{11} + C_{12} \right) \left( C_{33} - 2C_{13}^2 \right) \right) \right. \\
\left. \left. \left( 2N_1 + N_2 \right) N_2^2 \right) - \left( 2C_{13}^2 \left( 2N_1 + N_2 \right) N_2^2 + 4C_{13}C_{44}N_3^2 N_2 + (C_{11} + C_{12}) \left( 2N_1 + N_2 \right) C_{33}N_2^2 + C_{44}N_3^2 \right) \right] \right]. \\
(A.17) \]
\[ V_I = \frac{C_{11} + C_{12}}{2C_{13} - (C_{11} + C_{12})C_{33}} N_2 \left( \left( \dot{B} + \dot{Y} + \dot{\phi}_1 - \dot{\phi}_2 - C_{33}N_2^2 - C_{44}N_2^3 \right) + 2C_{13}N_2^2 + C_{44}N_3 \dot{M} \right) \]
\[ + \frac{C_{11} + C_{12}}{(C_{11} + C_{12})C_{33} - 2C_{13}^2 N_2^2} \times \left( 2C_{44}N_3 \dot{M} \left( \left( \dot{B} + \dot{Y} - \dot{\phi}_1 - \dot{\phi}_2 \right) (2C_{13}N_2 + (C_{11} + C_{12})(2N_1 + N_2)) \right) \right) \]
\[ + (4C_{13}N_2 + (C_{11} + C_{12})(2N_1 + N_2))C_{44}N_2^2 \left( (C_{11} + C_{12})C_{33} - 2C_{13}^2 \right) N_2^2 - \left( (B + Y - \dot{\phi}_1 - \dot{\phi}_2) (2C_{13}N_2 + (C_{11} + C_{12})(2N_1 + N_2)) \right) \]
\[ (2C_{13}N_2 + (C_{11} + C_{12})(2N_1 + N_2)) + (4C_{13}N_2 + (C_{11} + C_{12})(2N_1 + N_2))C_{44}N_2^2 + \left( (C_{11} + C_{12})C_{33} - 2C_{13}^2 \right) (2N_1 + N_2)N_2^2 - C_{44}M^2 \left( -2C_{13}^2 (2N_1 + N_2)^2 N_2^2 + 4C_{13}C_{44}N_2^2 N_2 + (C_{11} + C_{12})(2N_1 + N_2) \left( C_{33}N_2^2 + C_{44}N_2^3 \right) \right) \].

(A.18)

References


