Jump phenomena of rotational angle and temperature of NiTi wire in nonlinear torsional vibration

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A B S T R A C T
This paper performs experimental and analytical investigations on jump phenomena of thermomechanical responses of a nonlinear torsional vibration system with a phase transformable NiTi Shape Memory Alloy (SMA) wire as a nonlinear spring. By synchronizing the measurement of rotational angle and temperature of the NiTi wire in the torsional vibration system under external sinusoidal excitation, their evolutions in the transient and steady states are acquired. By monotonically increasing/decreasing the excitation frequency with small intervals around the primary resonance frequency, Frequency Response Curves (FRCs) of these two physical quantities are obtained and their jumps on the FRCs are captured in the frequency domain. To understand and quantify such jump phenomena, an effective Duffing oscillation model is introduced to simulate the mechanical behaviour of NiTi SMA in the dynamic system and to determine the FRCs and the characteristic jump frequencies. A lumped heat transfer analysis is then performed to establish the relationship between the temperature oscillation and the rotational angle. It is shown that the jump of the rotational angle is accompanied by a simultaneous jump of the temperature oscillation of the NiTi wire in torsional vibration. Particularly, the coupling effect between the thermal response and mechanical response can be significant and must be considered in the study of a nonlinear dynamic system with NiTi SMA component.

1. Introduction
NiTi Shape Memory Alloy (SMA) is attracting growing academic and industrial interests due to its unique superelastic and shape memory properties that originate from the reversible martensitic phase transition (Otsuka and Wayman, 1999). The superelasticity of SMA enables the recovery of large strain (~7%) with little residual deformation upon unloading. As a good candidate material for damping applications, SMAs have been used in civil engineering, vibration control and seismic devices (Dolce and Cardone, 2001a,b; Dolce et al., 2000; Heller et al., 2009; Saadat et al., 2002; Schmidt and Lammering, 2004; Van Humbeeck, 2003; Van Humbeeck and Kustov, 2005).

The stress-induced cyclic phase transition of NiTi SMA involves two intrinsic internal heat sources: latent heat (due to forward/reverse phase transition) and hysteresis heat (due to hysteretic constitutive relation) (He et al., 2010; Ortín and Delaey, 2002; Shaw and Kyriakides, 1995; Van Humbeeck, 2003; Yin et al., 2014). The generated heat will exchange with the ambient through heat convection and heat conduction, resulting in the temperature variation of the material. Since the phase transition stress depends on temperature according to the Clausius–Clapeyron relation (Rao and Rao, 1978), the above temperature variation will in turn lead to the variation of the stress–strain relation. Therefore, the stress–strain relation and the temperature oscillation are intrinsically coupled with each other and keep changing until reaching their respective steady state in the cyclic phase transition. Recent experimental investigations (Yin et al., 2014; Yin and Sun, 2012) investigated the coupling effect between the stress–strain relation and the temperature oscillation in the cyclic phase transition of NiTi SMA, both of which were significantly influenced by the deformation frequency. The hysteresis heat changes the mean temperature of the specimen while the latent heat is responsible for the temperature oscillation. Due to the thermomechanical coupling, the amount of stress drift reached 120 MPa in the transient state and the temperature oscillation in the steady state reached 23.0 °C when a NiTi bar was fully transformed at the deformation frequency of 1 Hz (see Fig. 1). Considering such significant coupling effects between the stress–strain relation and temperature oscillation, both the mechanical and thermal aspects of NiTi SMA should be considered in parallel when studying its cyclic behaviour.
Motivated by academic interests and industrial demands, some preliminary work has been done to study the dynamic behaviour of NiTi SMA in vibration system in the past two decades. Feng and Li (1996) and Li and Feng (1997) first measured transmissibility curves of SMA rod under traction-compression and bending. The resonance frequencies, peak responses and damping ratios were found to depend excitation amplitudes. Based on an improved Müller-Achenbach model, Seelecke (2002) studied the isothermal and non-isothermal dynamic responses of thin-walled SMA tubes in free and forced vibration systems by numerical simulations. With isothermal and non-isothermal dissipation functions, Bernardini and Rega (2005, 2010), Bernardini and Vestroni (2003), Lacarbonara et al. (2004) modelled the Frequency Response Curves (FRCs) of nonlinear shape memory oscillators and observed typical softening behaviours including the jump phenomenon, pitchfork and period-doubling. Machado and Lagoudas (2006), Machado et al. (2007a,b) investigated the dynamics and chaos of a nonlinear oscillator with SMA component by numerical simulations and experiments. Recently, Doaré et al. (2012) and Sbarra et al. (2011) conducted an experimental study on the quasi-static and dynamic torsional behaviour of NiTi wire. Though the jump phenomenon of rotational angle on the FRC was observed in experiments, thermal aspect of the dynamic system and the thermo-mechanical coupling effects were not considered. All of these previous theoretical and experimental works provide a starting point for the study of the thermomechanical responses of NiTi SMA in a dynamic system, particularly their jump phenomena on the FRCs.

In this paper, we present the experimental and analytical work on jumps of the rotational angle and temperature of NiTi wire in nonlinear torsional vibration. Section 2 describes basic properties of the material, experimental setup and synchronized measurement of the thermomechanical responses. Section 3 illustrates the effects of thermo-mechanical coupling on the dynamic response in the time domain. Jump phenomena of the rotational angle and
temperature and the effect of excitation amplitude are reported in this section. A simplified, effective Duffing oscillation model and a lumped heat transfer analysis are introduced in Section 4 to model the mechanical and thermal responses of the nonlinear torsional vibration system. The experimental and analytical work of this paper is summarized in Section 5 with key conclusions.

2. Sample preparation and experimental setup

2.1. Material and sample preparation

A polycrystalline superelastic NiTi wire (diameter \( D = 1.63 \) mm, total length \( L_{\text{total}} = 65.00 \) mm and gauge length \( L = 25.00 \) mm from Nitinol, USA) is used as the specimen in the dynamic tests as shown in Fig. 2(a). Basic chemical, mechanical and thermal properties of the material are listed in Table 1. Room temperature (\( T_0 \)) is maintained at \( T_0 \approx 220 \) °C during the experiments and therefore superelasticity of the specimen can be guaranteed.

Before the dynamic tests, the specimen was trained by two-way torsional loading–unloading under quasi-isothermal conditions for 200 cycles (see Fig. 2(b)) to remove the cyclic plasticity and to ensure that the torque–angle relation is repeatable. As shown in Fig. 2(c), in the frequency range of 0–800 Hz, the differences of the steady state torque–angle relations of the specimen at different typical frequencies are negligible in pure torsion. Thus, pure torsion with the rotational angle \( \alpha \) of \( \alpha = \pm 15^\circ/\pm 30^\circ/\pm 45^\circ/\pm 60^\circ/\pm 75^\circ \) are performed only at \( f = 0.700 \) Hz to characterise the steady state torque–angle relation of the specimen as shown in Fig. 2(d), where the isothermal torque–angle relation is also plotted in dotted line for comparison. It should be noticed that, different from the thin-walled tubes, due to the shear stress and shear strain inhomogeneities over the circular cross section of the wire, there is always a small residual rotation (\(<3^\circ\)) at zero torque during the cyclic torsion after training.

2.2. Experimental setup

The dynamic tests were performed on a MTS 858 Universal Testing Machine with specially designed grips to avoid possible sliding as schematically shown in Figs. 2(a) and 3(a). The upper grip was connected to a hydraulically driven rotational motor and the lower grip was attached to a thick stainless steel disc (mass \( m = 5.544 \) kg and rotational momentum of inertia \( J = 0.017 \) kg m\(^2\)) with a horizontal screw connexion. A single direction thrust ball bearing was used to hold the disc at the bottom and to cancel the axial force on the wire. An external sinusoidal angular excitation \( (\varphi = \frac{1}{2}(1 - \cos 2\pi ft)) \) was applied to the upper end of the specimen in the dynamic tests.

The rotational angle of the NiTi wire \( (\theta) \) is taken as the mechanical response of the torsional vibration system, which is the difference between the rotational angle of the disc \( (\theta_{\text{disc}}) \) and the external sinusoidal excitation \( (\varphi) \). \( \theta_{\text{disc}} \) was measured.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Chemical, mechanical and thermal properties of the material.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemical composition (wt%)</td>
<td>Ni 58.30%</td>
</tr>
<tr>
<td></td>
<td>Ti 41.70%</td>
</tr>
<tr>
<td>Austenite finish temperature</td>
<td>( A_f ) 18.51 °C</td>
</tr>
<tr>
<td>Shear modulus of NiTi in austenite phase</td>
<td>( G ) 14.5 GPa</td>
</tr>
<tr>
<td>Shear strain at the start of phase transition</td>
<td>( \gamma^0 ) 1.10%</td>
</tr>
<tr>
<td>Maximum phase transition shear strain</td>
<td>( \gamma^p ) 10.00%</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>( k ) 18.3 W/K m</td>
</tr>
<tr>
<td>Specific latent heat</td>
<td>( l_o ) 7.74 \times 10^7 J/m(^3)</td>
</tr>
<tr>
<td>Specific heat capacity</td>
<td>( \lambda = \rho C ) 3.225 \times 10^5 J/m(^3) K</td>
</tr>
</tbody>
</table>

Fig. 2. (a) Dimension of the specimen and clamping with grips, (b) evolution of the torque–angle relation of the specimen during pre-test training under \( \alpha = \pm 80^\circ \) and \( f = 0.005 \) Hz in pure torsion, (c) steady state torque–angle relation of the specimen under \( \alpha = 75^\circ \) and \( f = 0.650/0.700/0.750/0.800 \) Hz in pure torsion, (d) steady state torque–angle relation of the specimen under \( \alpha = \pm 15^\circ/\pm 30^\circ/\pm 45^\circ/\pm 60^\circ/\pm 75^\circ \) and \( f = 0.700 \) Hz in pure torsion.
Fig. 3. (a) Schematic drawing of the dynamic testing platform and (b) illustrative figure of the experimental configuration.

Fig. 4. Typical position images of the red mark on the disc during vibration. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 5. (a) Evolutions of $\theta$ and $T$ under the excitation of $A = 10^6$ and $f = 0.710$ Hz in the time domain, (b) steady state oscillation of $\theta$ and $T$, (c) steady state torque–angle relation of the specimen under $\alpha = \pm 24^\circ$ and $f = 0.710$ Hz in pure torsion.
Fig. 6. (a) Evolutions of $\theta$ and $T$ under the excitation of $A = 12^\circ$ and $f = 0.710$ Hz in the time domain, (b) steady state oscillation of $\theta$ and $T$, (c) steady state torque–angle relation of the specimen under $\alpha = \pm 61^\circ$ and $f = 0.710$ Hz in pure torsion.

Fig. 7. (a) Evolutions of $\theta$ and $T$ under the excitation of $A = 12^\circ$ and $f = 0.790$ Hz in the time domain, (b) steady state oscillation of $\theta$ and $T$, (c) steady state torque–angle relation of the specimen under $\alpha = \pm 42^\circ$ and $f = 0.790$ Hz in pure torsion.
via the Digital Image Correlation (DIC) technique from the videos recorded by a Canon 600D camera at 60 Frames Per Second (FPS). The cylindrical surface of the disc was marked by a vertical red line on the edge as shown in Fig. 4. \( \theta_{\text{disc}} \) was thus calculated from the recorded position history of the red mark on the disc using trigonometric functions.

The surface temperature of the NiTi wire was measured by a K-type CHAL thermocouple (diameter \( d = 25 \mu m \) from OMEGA, USA).
with a data logger (Agilent 34970A from Keysight, USA). In the frequency range of 0.650 Hz ~ 0.800 Hz, the characteristic heat conduction time \( t_h = \frac{D}{C} = 0.12 \text{ s} \) along the radius direction of the NiTi wire was much shorter than the deformation time \( t_\varepsilon = \frac{1}{f} \in [1.25 \text{ s}, 1.54 \text{ s}] \). The temperature field over the cross section can be regarded as uniform, so the surface temperature \( T \) measured by the thermocouple can well represent the volume average temperature of the wire and is used as the thermal response of the nonlinear torsional vibration system (He and Sun, 2011; Yin et al., 2014).

Using the shear modulus \( G \) of the NiTi wire in austenite phase, primary resonance frequency of the torsional vibration system in the linear elastic region can be estimated as

\[
f_0 = \frac{1}{2\pi} \sqrt{\frac{\pi GD^4}{32L}} \approx 0.774 \text{ Hz}
\]  

with \( G = 14.5 \text{ GPa} \), \( L = 25.00 \text{ mm} \), \( D = 1.63 \text{ mm} \) and \( f = 0.017 \text{ kg} \cdot \text{m}^{-2} \).

In order to observe the jump phenomena in the frequency domain, we set the excitation frequency \( f \) around \( f_0 \) in the range of \( f \in [0.650 \text{ Hz}, 0.800 \text{ Hz}] \) with 0.005 Hz interval under 5 excitation amplitudes \( A = 4/6/8/10/12 \). At each excitation amplitude, dynamic tests were performed at 31 given frequencies according to the ascending order (from 0.650 Hz up to 0.800 Hz at 0.005 Hz interval) and descending order (from 0.800 Hz down to 0.650 Hz at 0.005 Hz interval) respectively. A total of 310 dynamic tests were performed in this paper.

### 3. Experimental results and discussions

We first give the general features of the thermomechanical responses of the nonlinear dynamic system in the time domain (\( \theta \sim t \) and \( T \sim t \) relations) in Section 3.1. FRCs of both rotational angle (\( \Delta \theta \sim f \) relation) and temperature oscillation (\( \Delta T \sim f \) relation) are presented in Section 3.2, where the jump frequencies \( f_{\text{up}} \) and \( f_{\text{down}} \) and jump amplitudes \( \Delta \theta_{\text{up}}, \Delta \theta_{\text{down}}, \Delta T_{\text{up}} \) and \( \Delta T_{\text{down}} \) are identified. In Section 3.3, the effects of excitation amplitude \( A \) on the FRCs and the jump phenomena are reported.

#### 3.1. Typical thermomechanical responses in the time domain

Fig. 5(a) shows the evolutions of \( \theta \) and \( T \) with time under the external excitation of \( A = 10^\circ \) and \( f = 0.710 \text{ Hz} \). It is seen that both \( \theta \sim t \) and \( T \sim t \) relations consist of a transient state and a steady state. In the steady state, \( \theta \) oscillates around 0° symmetrically with \( \Delta \theta = 48^\circ \) and \( T \) oscillates around \( T_{\text{mean}} = 22.42 \text{ °C} \) with \( \Delta T = 0.18 \text{ °C} \). As shown in Fig. 5(c), under \( \alpha = \pm 24^\circ \) and \( f = 0.710 \text{ Hz} \) in pure torsion, the specimen is mainly in the linear elastic stage and little phase transition happens, resulting in a small temperature oscillation. Meanwhile, as there is little hysteresis heat release and accumulation in the dynamic test, \( T_{\text{mean}} \) does not change much with time and the observed small fluctuation of the mean temperature in the steady state is due to the uncertainty of room temperature.

Fig. 5(a) shows the thermomechanical responses of the wire under the excitation of \( A = 12^\circ \) and \( f = 0.710 \text{ Hz} \). Compared with
due to the hysteresis heat absorption in the two-way torsional response. From the corresponding torque–angle relation of the specimen under $\alpha = \pm 61^\circ$ and $f = 0.710$ Hz in Fig. 6(c), it is seen that much more phase transition is involved in the steady state oscillation, which is accompanied with more latent heat release/absorption and hysteresis heat accumulation. Consequently, there is a significant increase of $T_{\text{mean}}$ in the transient state and $\Delta T$ is much larger in the steady state.

Fig. 7(a) shows the $\theta \sim t$ and $T \sim t$ relations when $A = 12^\circ$ and $f = 0.790$ Hz. With the same excitation amplitude but an increased frequency, both $\Delta \theta$ and $\Delta T$ in the steady state are reduced compared with Fig. 6(a). This is mainly because $f$ deviates from the primary resonance frequency, which is similar to the response of a linear vibration system and will be discussed in detail in Section 3.2.

To demonstrate the effect of thermomechanical coupling on the nonlinear vibration responses (particularly the influence of temperature on the mechanical response), dynamic tests under non-isothermal and quasi-isothermal conditions were performed at the same external excitation. By introducing air flow (with compressed air) as shown in Fig. 8(b), the heat convection coefficient was enhanced and a quasi-isothermal testing condition was realised in the experiments. It is seen that compared with the case in still air in Fig. 8(a), $\Delta T$ in the steady state reduces significantly from 1.32 °C to 0.35 °C and $T_{\text{mean}}$ almost does not change with time in the transient state. At the same time, the evolution of $\theta$ differs a lot and $\Delta \theta$ in the steady state increases from 81.05° to 106.34°. To have a better understanding, the isothermal shear stress and shear strain ($\tau - \gamma$) relations of a superelastic NiTi tube at three different ambient temperatures are shown in Fig. 8(c). It is seen that the increase of the ambient temperature will reduce the amount of shear strain for a given shear stress (e.g., $\tau = 300$ MPa) in pure torsion. Similarly, for the dynamic response under non-isothermal condition in Fig. 8(a), the increased $T_{\text{mean}}$ due to the hysteresis heat accumulation will decrease the $\Delta \theta$ compared with the quasi-isothermal case in Fig. 8(b). The effect of heat transfer condition on $\Delta T$, $T_{\text{mean}}$ and $\Delta \theta$ in the dynamic system are clearly demonstrated by the comparison.

### 3.2. Steady state thermomechanical responses in the frequency domain under $A = 10^\circ$

At a given excitation amplitude of $A = 10^\circ$, a total of 62 vibration tests were performed at 31 given excitation frequencies. The obtained $\Delta \theta$ and $\Delta T$ from the steady state oscillations under each
and away from the primary resonance frequency, which increases, both magnitudes of rotational angle (\( \Delta \theta \)) and temperature (\( T_{\text{mean}} \)) relations from all the dynamic tests. For the nonlinear torsional vibration system, when the excitation amplitude is increased to 10\(^\circ\) or 12\(^\circ\) (see Fig. 10, Fig. 11(d) and (e) for the torque–angle relations), \( \Delta \theta \) and \( T_{\text{mean}} \) increase more significantly and the jump phenomena becomes more obvious.

In summary, the thermomechanical responses of the dynamic system in the time domain are presented and the jumps of the steady state \( \Delta \theta \) and \( T_{\text{mean}} \) on the FRCs are identified under different excitation amplitudes in this section. When the phase transition is induced by shear stress at large rotation, the torque–angle relation of the specimen becomes nonlinear and hysteretic. Such softening-type nonlinearity is responsible for the observed jump phenomenon on the FRC of \( \Delta \theta \). The release/absorption of the latent heat during forward/reverse phase transition during loading/unloading of the specimen will lead to the temperature oscillation (\( \Delta T \)), while the accumulation of hysteresis heat will increase the mean temperature \( (T_{\text{mean}}) \) as shown in Fig. 12. With such preliminary interpretation of the experimental observations, the theoretical modelling is performed in Section 4 to quantify the thermomechanical jump phenomena on the FRCs.

**4. Modelling and analysis**

It is well known that the jump phenomenon on the FRC is essentially due to the nonlinearity of the spring in a nonlinear vibration system. The effect of nonlinearity on the dynamic response is well and most simply described by the Duffing oscillation model consisting of a cubic nonlinear spring and a viscous damper. In this paper, the basic assumption is that there is similarity between a Duffing oscillator and the dynamic system with NiTi SMA, in which the observed jump phenomena essentially originates from the nonlinearity of NiTi SMA during the reversible martensitic phase transition. From the theoretical point of view, the nonlinear hysteretic torque–angle relation of NiTi wire can be directly used to simulate the jump phenomena but only numerically. On the other hand, the Duffing oscillator is the simplest nonlinear model to capture the key features of the jump phenomenon and to give an approximate but analytical solution of the FRC. The validity of using the Duffing oscillation model, including the parameter identifications, to quantify the jumps of the thermomechanical response of the NiTi wire is justified based on the following.

(1) The Duffing oscillation model only describes the mechanical response of a nonlinear dynamic system in the steady state. However, for the dynamic system with NiTi wire, both the mechanical response and thermal response should be quantified by considering the thermomechanical coupling effect.
In this paper, the mechanical response and thermal response are treated separately in the following theoretical analysis. The steady state mechanical response is modelled by a Duffing oscillator and the thermal response is then determined by a lumped heat analysis based on the obtained mechanical response. The thermomechanical coupling effect is simplified by such decoupling so as to capture the key features, i.e., the rotational angle determines the temperature oscillation and there is no vice versa.

(2) Different from the nonlinear elastic spring in a Duffing oscillator, the response of the NiTi wire in the dynamic system is the response of a mechanical structure (long solid cylinder) and its stiffness is not a purely material constitutive property because the phase transition is not homogeneous and the shear stress is not uniform over the circular cross section. However, the NiTi wire can still be treated as an “effective nonlinear spring” with its nonlinear torque–angle relation in the same spirit as the Duffing oscillator. In determining the nonlinear vibration response, all the parameters are directly identified from the nonlinear torque–angle relation instead of from the constitutive relation of NiTi SMA. In this way, the non-uniform shear stress and shear strain over the cross section will not directly get involved in the modelling. In quantifying the thermal response, a simplified “elastoplastic” phase transition model is used to calculate the distribution of transformation shear strain and to determine the total amount of heat release/absorption during forward/reverse phase transition. Thus the temperature oscillation can be predicted by the lumped heat analysis.

(3) In a Duffing oscillator, the nonlinear spring and the viscous damper are separate and independent. However, for the NiTi SMA, both the nonlinear stiffness and the damping capacity come from the nature of the first-order phase transition and manifest macroscopically as the hysteretic torque–angle relation. The stiffness and the damping capacity can affect each other via the thermomechanical coupling. For the purpose of simplicity, the softening stiffness and the hysteresis loop of the torque–angle relation of NiTi wire are separately modelled as an “effective nonlinear spring” and an “effective viscous damper” in the following theoretical analysis.

(4) The phase transition will decrease the stiffness of the NiTi wire compared with that of the initial austenite phase. Before the NiTi wire is fully transformed, its stiffness decreases monotonically with the rotational angle. The slight nonsymmetry of the torque–angle relation in two-way torsion in Fig. 2(c) and (d) is ignored and the stiffness during forward and reverse rotation is also treated as the same in this paper. Consequently, the stiffness of NiTi wire can be approximated by a polynomial equation with a linear and a cubic term as the nonlinear spring in a Duffing oscillator. Meanwhile, the hysteretic effect is equivalently modelled by the “effective viscous damper”, which dissipates the same amount of energy per cycle as the hysteresis loop of the NiTi wire. As shown later in this paper, such simplification and approximation of the NiTi wire as a cubic
nonlinear spring and a viscous damper can capture the key features of the experimental observations of the jump phenomena in the frequency domain.

Based on the above justifications, the Duffing oscillation model with a cubic nonlinear spring and a viscous damper can now be used to quantify the mechanical response of a nonlinear dynamic system with NiTi wire. After obtaining the mechanical response, the lumped heat analysis further determines the latent heat release/absorption in the forward/reverse phase transition and predicts the temperature oscillation amplitude in the steady state.

4.1. Theoretical analysis of rotational angle using effective Duffing oscillation model

The dynamic system with nonlinear hysteretic NiTi SMA in Fig. 13(a) can be modelled as an effective Duffing oscillator with a nonlinear spring and a viscous damper. In the effective modelling, the nonlinearity and dissipative property of the NiTi wire with partial phase transition are separately modelled as an equivalent cubic nonlinear spring \( M_{\text{spring}} = k_1 \theta + k_2 \theta^3 \) and an equivalent angular viscous damper \( M_{\text{damper}} = c_\theta \dot{\theta} \) in Fig. 13(b).

The equation of motion of the disc can be expressed as,

\[
\ddot{\theta} + c_\theta \dot{\theta} + k_1 \theta + k_2 \theta^3 = -2A\pi^2 f^2 \cos 2\pi ft
\]

where \( \theta_{\text{disc}} \) is the rotational angle of the disc and \( \dot{\theta} \) (\( = \theta_{\text{disc}} - \varphi \)) is the rotational angle of the wire.

Replacing \( \theta_{\text{disc}} \) by \( \theta \) and \( \varphi = \frac{\pi}{2}(1 - \cos 2\pi ft) \), Eq. (4.1) becomes

\[
\ddot{\theta} + c_\theta \dot{\theta} + k_1 \theta + k_2 \theta^3 = -2A\pi^2 f^2 \cos 2\pi ft
\]

Eq. (4.2) describes a torsional vibration system with a cubic nonlinear spring \( M_{\text{spring}} = k_1 \theta + k_2 \theta^3 \) and an equivalent viscous damper \( M_{\text{damper}} = c_\theta \dot{\theta} \) under a sinusoidal torque excitation \( M = -2A\pi^2 f^2 \cos 2\pi ft \) as shown in Fig. 13(c).

In the effective modelling, the nonlinear spring constants \( k_1 \) and \( k_2 \) and the damping coefficient \( c_\theta \) can vary with the rotational angles in the transient state. For the purpose of simplicity, we only consider the steady state response of the dynamic system, where \( c_\theta, k_1 \) and \( k_2 \) can be approximated as constants. The steady state solution of Eq. (4.2) is obtained by solving the primary resonance for a small cubic nonlinear dynamic system with the method of multiple scales (Fidlin, 2005; Nayfeh, 2011; Nayfeh and Mook, 2008). Assuming the steady state solution is a sinusoidal periodic function with frequency \( f \) at phase angle \( \chi \).

\[
\theta = \frac{\Delta \theta}{2} \cos(2\pi ft + \chi)
\]

The steady state frequency-response equation is then given by,

\[
\left[ \frac{\Delta \theta}{2} \left( k_1 - 4\pi^2 f^2 J + \frac{3}{2} k_2 \left( \frac{\Delta \theta}{2} \right)^2 \right) \right]^2 + \left( -\pi c_\theta \Delta \theta \right)^2 = (2A\pi^2 f^2)^2
\]

Eq. (4.3) is an implicit solution of rotational angle amplitude \( \Delta \theta \) as a function of the external excitation \( f \) and \( A \), equivalent spring constants \( k_1 \) and \( k_2 \) and effective damping coefficient \( c_\theta \). The FRCs \( (\Delta \theta \sim f) \) relation under given excitations can be determined from this equation. Meanwhile, the “backbone” curve (FRC of undamped free Duffing oscillation) can be determined from Eq. (4.3) with \( c_\theta = A = 0 \).

\[
\Delta \theta = 4\sqrt{\frac{\pi^2 f^2 J - k_1}{3k_2}}
\]

For the nonlinear torsional vibration system with a NiTi wire, the material parameters in Eq. (4.3) and Eq. (4.4) are determined as: \( J = 0.017 \text{ kg} \cdot \text{m}^2 \), \( c_\theta = c_\theta = 3.503 \times 10^3 \text{ N} \cdot \text{m} \cdot \text{s} \), \( k_1 = 0.400 \text{ N} \cdot \text{m} \) and \( k_2 = -0.090 \text{ N} \cdot \text{m} \) (see Appendix for details). The external excitation parameters are: \( A \in [4^\circ:12^\circ] \) with \( 2^\circ \) interval and \( f \in [0.650 \text{ Hz}:0.800 \text{ Hz}] \) with 0.005 Hz interval.

Fig. 14(a) shows the FRCs from Eq. (4.3) under excitation amplitudes of \( A = 4^\circ:6^\circ:8^\circ:10^\circ:12^\circ \) when \( c_\theta = 3.503 \times 10^3 \text{ N} \cdot \text{m} \cdot \text{s} \). Fig. 14(b) shows the simulated FRCs with different equivalent damping coefficients of \( c_\theta = 0/0.002/0.004/0.006/0.008 \text{ N} \cdot \text{m} \cdot \text{s} \).
(a) Linear elastic deformation (at small rotation)
(b) “Elastoplastic” deformation (at large rotation)
(c) “Elastoplastic” τ – γ relation with perfect transformation plasticity.

![Fig. 16](image)

Fig. 16. (a) Small strain linear elastic deformation of the specimen at small rotational angle, (b) small strain “elastoplastic” deformation of the specimen at large rotational angle and (c) illustrative “elastoplastic” τ – γ relation of the NiTi material with perfect transformation plasticity.

(a) Temperature oscillation vs. Rotational angle

![Fig. 17(a)](image)

(b) Temperature oscillation vs. Frequency

![Fig. 17(b)](image)

Fig. 17. (a) Theoretical prediction and experimental data of the relationship between ΔT and Δθ. (b) comparison of the FRCs of ΔT between theoretical prediction and experimental data under A = 4°/6°/10°/12°.

when A = 6°. It is seen that all of the FRCs lean to the left. As the excitation amplitude (A) increases, the jump frequencies (f_{up} and f_{down}) consistently shift to the left and the jump magnitudes (Δθ_{up} and Δθ_{down}) increase. Meanwhile, the equivalent damping coefficient (c_{eq}) can significantly affect the bandwidth of the FRCs. These are all key features of the FRCs of a softening-type nonlinear dynamic system.

The comparison between the theoretical predictions and the experimental data of the FRCs, the characteristic frequencies (f_{up} and f_{down}) and the jump magnitudes (Δθ_{up} and Δθ_{down}) are shown in Fig. 15. It is seen that the general features of the mechanical response in the frequency domain predicted by the Duffing oscillation model agree well with the experimental data. The data from modeling are larger than the experiments, especially at small excitation amplitudes (A = 4° and 6°). This might be due to the external energy dissipation of the mechanical system (e.g., friction from the bearing and air damping). Meanwhile, the experimental data shows that the nonlinearity increases with the excitation amplitude; but in the modeling the nonlinearity of the material is overestimated for small rotations in the linear elastic stage where there is little phase transition.

Fig. 15(b) shows the comparison of the jump-up frequency (f_{up}), the jump-down frequency (f_{down}) and the frequency gap (Δf) between the experimental data and modeling under different excitation amplitudes. It is seen that the theoretical prediction of f_{up} (in black circle) are quite close to the experimental data (in black triangle), while the predicted f_{down} (in red circle) are smaller than experimental data (in red triangle). This might be because the metastable state with large rotational angle cannot be further maintained at lower frequencies following the descending frequency order. Any slight external disturbance can make the jump-down phenomenon take place earlier than the ideal theoretical prediction in the frequency domain. Consequently, the frequency gaps from the experiments (in blue triangle) are smaller than the modelling (in blue circle). Although there are some quantitative errors, the overall trends of f_{up}, f_{down} and Δf agree qualitatively well with the experimental data.

Fig. 15(c) shows the comparison of the jump magnitudes (Δθ_{up} and Δθ_{down}) between the modelling and experiments under different excitation amplitudes. The jump-up magnitudes from theoretical modeling (in black circle) are very close to the experimental data (in black triangle) but jump-down magnitudes (in red circle) are larger than the experimental data (in red triangle). Both the modeling and experiments show that the jump magnitudes increase with the excitation amplitude.

From the comparison and discussion above, it is seen that although there are some quantitative errors, the simplified Duffing oscillation model can well characterize the softening-type nonlinear behaviour of NiTi SMA in a dynamic system and capture the key features of jump phenomenon of the rotational angle on the FRC.

4.2. Theoretical analysis of temperature oscillation using lumped heat analysis

To quantify the jump phenomenon of the steady state temperature oscillation, a lumped heat transfer analysis is performed to calculate the temperature oscillation (ΔT) at different rotational angles (Δθ).

ΔT(°C)

A=4° modelling
A=6° modelling
A=8° modelling
A=10° modelling
A=12° modelling
A=4° experiment
A=6° experiment
A=8° experiment
A=10° experiment
A=12° experiment

ΔT(°C)

0.65 0.70 0.75 0.80

f(Hz)

Δθ(°)

0 30 60 90 120 150

ΔT(°C)

0 1 2 3 4

Δθ(°)

0 30 60 90 120 150

ΔT(°C)

0 1 2 3 4

f(Hz)
As shown in Fig. 16(a), the entire cross section is in the linear elastic stage at small rotation and there is no phase transition, no latent heat and no hysteresis heat release. The maximum rotational angle (\(\gamma_c\)) of such pure linear elastic stage is determined by

\[ \gamma_c = \frac{2l_1l_2}{D} \]  

(4.5)

where \(\gamma_c(=1.10\%)\) is the shear strain at the start of phase transition.

When \(\gamma > \gamma_c\), phase transition starts from the cylindrical surface of the NiTi wire and the distribution of the shear stress and shear strain over the cross section are shown in Fig. 16(b). For the purpose of simplicity, we ignore the small hardening of the NiTi material (Li, 2002; Sun and Li, 2002) and approximate the shear behaviour as “elastoplastic” type with perfect transformation plasticity (Fischer et al., 1996; Hill, 1998; Mendelson, 1968) in Fig. 16(c).

In the linear elastic region, both the shear stress and shear strain are linear proportional to \(r\). In the phase transition region, the shear stress keeps a constant (\(\tau = \tau_c\)) and the shear strain (\(\gamma(r)\)) is still proportional to \(r\) and is the combination of maximum elastic shear strain (\(\gamma_c\)) and phase transition shear strain (\(\gamma^{PT}(r)\)).

\[ \gamma(r) = \gamma_c + \gamma^{PT}(r) \]

For partial phase transition (\(\gamma^{PT}(r) < \gamma_0^{PT}\)) in this paper, the released latent heat \(L_r(r)\) at rotational angle of \(\pi/2\) and radius of \(r\) is proportional to the phase transition shear strain \(\gamma^{PT}(r)\) as

\[ L_r(r) = \frac{\gamma^{PT}(r)}{\gamma_c} L_0 = \frac{\gamma(r) - \gamma_c}{\gamma_c} L_0 \]

(4.6)

where \(\gamma_0^{PT}(=10.00\%)\) is the maximum phase transition shear strain.

The phase transition region is defined by \(\frac{\gamma_c}{\gamma} \leq r \leq \frac{\gamma_0^{PT}}{\gamma}\) and the total latent heat release per unit length of the wire is determined by the integral,

\[ \xi_2 = \int_{\xi_1}^{\xi_2} L_r(r) \cdot 2rdr \]

Ignoring the heat conduction, the average latent heat per unit volume of the wire is

\[ \xi_2 = \frac{\xi_2^\text{total}}{2\pi D^2} = \left( \frac{D}{3L_0^{\gamma_c}} \frac{\tau}{D^{\gamma_c}} + \frac{4L_1^2l_2}{3L_0^{\gamma_c}} \frac{1}{D^{\gamma_c}} \frac{\gamma_c}{\gamma} L_0 \right) \]

(4.7)

Due to the superelasticity, the latent heat release during loading is \(E_0\) and the latent heat absorption during unloading is \(-E_0\) (assuming full recovery of the \(\gamma^{PT}(r)\) during unloading). Using the analytical solution of temperature oscillation \(\Delta T\) in the cyclic phase transition (Yin et al., 2014; Yin and Sun, 2012), we have

\[ \Delta T = \frac{E_0}{\lambda^2 \left( 1 + \frac{1}{4 \pi^2 (\Delta T)^2} \right) \lambda^2} \]

In the dynamic tests, \(f_{ST} \in [1.0 \text{ Hz} \text{ to } 16.0 \text{ Hz}]\) (notice that there is a frequency-doubling of temperature oscillation) and \(\Delta T \approx 5 \text{ s} \) (He and Sun, 2011). Considering \(4\pi^2 (f_{ST})^2 \Delta T^2 \approx 1\) in this paper, the heat convection is ignored and \(\Delta T\) is approximated as

\[ \Delta T \approx \frac{E_0}{\lambda^2 \left( 1 + \frac{16\pi^2 (f_{ST})^2}{4 \pi^2 (f_{ST})^2} \frac{1}{(\Delta T)^2} \right) \lambda^2} \]

(4.8)

In the steady state oscillation, \(\lambda = \frac{1}{4} \Delta \theta\) and the amplitude of temperature oscillation \(\Delta T\) is thus expressed as

\[ \Delta T = \left\{ \begin{array}{ll} 0 & \lambda \leq \frac{4\pi}{\Delta \theta} \\ \frac{D}{\pi D} \Delta \theta + \frac{16\pi^2 (f_{ST})^2}{4 \pi^2 (f_{ST})^2} \frac{1}{(\Delta T)^2} \frac{\gamma_c}{\gamma} L_0 & \lambda > \frac{4\pi}{\Delta \theta} \end{array} \right. \]

(4.9)

With Eq. (4.9), the temperature oscillation amplitude \(\Delta T\) can now be plotted as a function of the rotational angle \(\Delta \theta\) in Fig. 17(a) and the experimental data are also plotted for comparison. It is seen that the theoretical prediction of \(\Delta T\) agrees very well with the experimental data. At small rotation, there is no phase transition and no heat sources involved, so \(\Delta T\) is zero. At large rotation, \(\Delta T\) is almost linear proportional to \(\Delta \theta\). Consequently, when phase transition is induced by shear stress in torsion, jump phenomenon of the rotational angle \(\Delta \theta\) on the FRC is always accompanied by a simultaneous jump of the temperature oscillation \(\Delta T\) in the frequency domain.

The FRCs of the temperature oscillation in Fig. 17(b) are obtained from Eqs. (4.3) and (4.9). The experimental data of temperature oscillation in the frequency domain are also plotted for comparison. It is seen that the FRCs of the thermal response under different excitation amplitudes have the same features as those of the mechanical response. Meanwhile, the general features of the FRC of \(\Delta T\) from the modelling agree with the experiments. However, due to the accumulated error from the theoretical prediction of the FRC of \(\Delta \theta\), there are some differences between the modelling

<table>
<thead>
<tr>
<th>Rotational angle ((\gamma))</th>
<th>Frequency (Hz)</th>
<th>Hysteresis heat (kJ/m)</th>
<th>Equivalent damping coefficient (c_0) (N m s)</th>
<th>Mean equivalent damping coefficient (\overline{c_0}) (N m s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 ((\gamma_{\text{surface}} = 0.85%))</td>
<td>0.650</td>
<td>3.328 x 10^{-3}</td>
<td>3.784 x 10^{-3}</td>
<td>3.503 x 10^{-3}</td>
</tr>
<tr>
<td>0.700</td>
<td>3.409 x 10^{-3}</td>
<td>3.600 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.750</td>
<td>3.324 x 10^{-3}</td>
<td>3.276 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.800</td>
<td>3.261 x 10^{-3}</td>
<td>3.013 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.650</td>
<td>1.411 x 10^{-2}</td>
<td>4.020 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.700</td>
<td>1.419 x 10^{-2}</td>
<td>3.746 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.750</td>
<td>1.389 x 10^{-2}</td>
<td>3.422 x 10^{-3}</td>
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</tr>
<tr>
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<td>1.352 x 10^{-2}</td>
<td>3.123 x 10^{-3}</td>
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</tr>
<tr>
<td>0.650</td>
<td>3.239 x 10^{-2}</td>
<td>4.092 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.700</td>
<td>3.231 x 10^{-2}</td>
<td>3.791 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.750</td>
<td>3.172 x 10^{-2}</td>
<td>3.473 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.800</td>
<td>3.100 x 10^{-2}</td>
<td>3.182 x 10^{-3}</td>
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<tr>
<td>0.850</td>
<td>5.675 x 10^{-2}</td>
<td>4.033 x 10^{-3}</td>
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<tr>
<td>0.700</td>
<td>5.605 x 10^{-2}</td>
<td>3.699 x 10^{-3}</td>
<td></td>
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</tr>
<tr>
<td>0.750</td>
<td>5.451 x 10^{-2}</td>
<td>3.358 x 10^{-3}</td>
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<tr>
<td>0.800</td>
<td>5.350 x 10^{-2}</td>
<td>3.089 x 10^{-3}</td>
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<tr>
<td>0.650</td>
<td>8.485 x 10^{-2}</td>
<td>3.860 x 10^{-3}</td>
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<tr>
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<tr>
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<td>3.148 x 10^{-3}</td>
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<tr>
<td>0.800</td>
<td>7.809 x 10^{-2}</td>
<td>2.886 x 10^{-3}</td>
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</table>
and experiments on the FRCs of $\Delta T$, especially at small excitation amplitudes ($A = 4^\circ$ and $6^\circ$).

5. Summary and conclusions

Mechanical and thermal responses of a NiTi SMA wire in nonlinear torsional vibration are examined in this paper. By synchronized measurement of rotational angle and temperature of the NiTi wire under external sinusoidal excitation in dynamic tests, both transient state and steady state thermomechanical responses were acquired in the excitation frequency range of $0.650\text{ Hz} - 0.800\text{ Hz}$ and the excitation amplitude range of $4^\circ$–$12^\circ$. Simultaneous jumps of the rotational angle and temperature oscillation of the wire were observed on the FRCs in the frequency domain. To understand such jump phenomena, a simplified effective Duffing oscillation model consisting of a softening-type cubic nonlinear spring and an equivalent viscous damper was introduced to determine the FRCs and quantify the observed vibration instability. A lumped heat transfer analysis was then performed to establish the relationship between the rotational angle and temperature oscillation of the NiTi wire in the steady state. Consequently, the FRCs and the jump phenomena of the temperature oscillation can be predicted by the modelling. The comparison between the theoretical modelling and experimental data shows that the Duffing oscillation model and lumped heat transfer analysis can capture the key qualitative features of the mechanical and thermal responses of a dynamic system with phase-transformable NiTi wire. Both the mechanical and thermal aspects of the nonlinear dynamic system with NiTi SMA were revealed and quantified. The main conclusions of this paper are the followings:

- The jump phenomenon of the mechanical response on the FRC is accompanied by a simultaneous jump of the thermal response in the frequency domain for a dynamic system with NiTi SMA component. The jump of the mechanical response originates from the softening-type nonlinearity in the phase transition of NiTi SMA while the jump phenomenon of the thermal response is due to the accompanied latent heat release/absorption in the forward/reverse phase transition. The thermal response is an indispensable aspect of a dynamic system with NiTi SMA, especially at large torsion of the wire.
- The thermomechanical coupling plays an important role in a nonlinear vibration system with NiTi SMA. The mechanical response will influence the thermal response through the involved two intrinsic heat sources (latent heat and hysteresis heat); the thermal response can in turn affect the mechanical response through the Clausius–Clapeyron relation. The significant coupling effects between the mechanical response and thermal response should be further considered in detail for a dynamic system with NiTi SMA.
- The external excitation amplitude can significantly influence the jump phenomena and the FRCs of a nonlinear dynamic system with NiTi SMA component. With the increase of excitation amplitude, the jump frequencies decrease, while the frequency gap and the jump magnitude of both mechanical and thermal responses increase.
- When NiTi SMA wire is used as nonlinear damping spring in a dynamic system with partial phase transition, the mechanical and thermal responses can be approximately quantified by an effective simplified Duffing oscillation model and a lumped heat transfer analysis respectively.

Acknowledgements

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Appendix. Determination of material parameters

For a superelastic NiTi wire, the energy dissipation in one loading–unloading cycle is the hysteresis loop area (He and Sun, 2011; He et al., 2010), which is equivalent to the integral of the torque–angle relation in one cycle.

$$D_{\text{hysteresis}} = \int M \cdot d\theta$$

For a mechanical vibration system, the energy dissipation of a viscous damper (with viscosity $c$) in one cycle (with rotational angle $\beta(t) = \alpha \sin 2\pi ft$) is,

$$W_{\text{damper}} = \int_0^{1/f} c\beta \cdot d\beta = \int_0^{1/f} c\beta^2 dt$$

The equivalent damping coefficient $(c_{eq})$ is obtained by equaling the steady state hysteresis loop area of NiTi SMA in cyclic deformation to the energy dissipation of an equivalent viscous damper in a mechanical vibration system (Rao, 2011) as

$$D_{\text{hysteresis}} = W_{\text{damper}}$$

Thus,

$$c_{eq} = \frac{D_{\text{hysteresis}}}{2\pi f^2\alpha^2} \quad \text{(A.1)}$$

By using the values of the steady state hysteresis loop area $(D_{\text{hysteresis}})$ of the specimen under different deformation frequencies ($f = 0.650/0.700/0.750/0.800\text{ Hz}$) and rotational angles ($\alpha = \pm 15^\circ/\pm 30^\circ/\pm 45^\circ/\pm 60^\circ/\pm 75^\circ$) in pure torsion, the equivalent damping coefficients $(c_{eq})$ are obtained from Eq. (A.1) and listed in Table 2. It is seen that the rate effect can be ignored under a given rotational angle and the difference of $c_{eq}$ is very small under different rotational angles. Consequently, the equivalent damping coefficient $(c_{eq})$ can be treated as a constant and approximated by the mean value.

$$c_{eq} = c_{eq} = 3.503 \times 10^{-3} \text{ N} \cdot \text{m} \cdot \text{s} \quad \text{(A.2)}$$
The linear and cubic spring constants \(k_1\) and \(k_2\) respectively in a Duffing oscillator are fitted from the steady state torque–angle relations of the wire to characterise the softening-type nonlinear behaviour in dynamic tests. \(k_1\) is proportional to the shear modulus \(\sigma_{00}\) respectively) in the measured steady state torque–angle relation under nonlinearity of the specimen or how “soft” the NiTi wire is.

The comparison between the cubic nonlinear fitting curve and the measured steady state torque–angle relation under \(\pm 30^\circ/\pm 45^\circ/\pm 60^\circ/\pm 75^\circ\) and \(f = 0.700 \text{ Hz}\) is shown in Fig. 18. It is seen that the cubic nonlinear fitting can well capture the behaviour of the NiTi wire as a nonlinear softening spring.

\[
k_1 = 0.400 \text{ N} \cdot \text{m} \\
k_2 = -0.090 \text{ N} \cdot \text{m}
\]

References


