Thermomechanical responses of nonlinear torsional vibration with NiTi shape memory alloy – Alternative stable states and their jumps

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\section*{ABSTRACT}

The dynamic response of nonlinear torsional vibration system with phase transformable NiTi Shape Memory Alloy (SMA) wire is investigated by experiment in this paper. The thermomechanical responses of the NiTi wire as a softening nonlinear damping spring in the torsional vibration system are measured by synchronized acquisition of rotational angle and temperature under external excitation. \textit{Frequency Response Curves} (FRCs) at fixed excitation amplitude and \textit{Amplitude Response Curves} (ARCs) at fixed frequency are obtained in the frequency and amplitude domains respectively. It is found that, as the deformation of NiTi wire goes into the softening nonlinear phase transition region, the smooth and stable dynamic responses along one branch of FRC or ARC will gradually enter into metastable region and eventually become unstable and drastically switch to a new contrasting alternative stable state along the other branch. The jump phenomenon between the alternative stable states on the lower and upper branches of the FRC or ARC and the hysteresis between the jump-up and jump-down are identified by experiments. In addition, the effects of external disturbance (both magnitude and direction) on triggering the jumps between the alternative stable states along the two metastable branches are examined in the time domain. The stability of the nonlinear dynamic response is analyzed by the Duffing oscillator model and interpreted via the stability landscape. For the first time, we directly reveal the alternative stable states and jump phenomena of thermomechanical responses by experiments in the frequency, amplitude and time domains. The results not only show the important roles of phase transition nonlinearity in bringing multiple equilibrium states and their fast switches, but also provide a solid experimental base for the identification of metastable regions as well as further management of the undesired dynamic responses of vibration system where NiTi is used as a nonlinear damping spring.

1. Introduction

Originating from the reversible, first-order martensitic phase transition, NiTi \textit{Shape Memory Alloy} (SMA) wire accommodates large recoverable deformation and has large damping capacity \cite{Otsuka1999}. The excellent thermomechanical properties make NiTi an excellent candidate material for vibration control and damping applications \cite{Lagoudas2008, Saadat2002}.
However, several great challenges still exist in the application of phase transformable NiTi SMA in vibration systems.

The first challenge lies in the strong thermomechanical coupling in the behavior of NiTi SMA as shown in Fig. 1. Two intrinsic internal heat sources exist in the stress-induced cyclic phase transition of superelastic NiTi: latent heat and hysteresis heat (Shaw and Kyriakides, 1995; Van Humbeek, 2003; Yin et al., 2014). The generated heat will exchange with the ambient through heat convection and conduction, resulting in a temperature variation of the material. The phase transition stress depends on temperature following the Clausius-Clapeyron relation (Rao and Rao, 1978) and therefore the temperature variation will change the stress-strain relation, which in turn influences the two internal heat sources. Consequently, the stress-strain relation and temperature oscillation are intrinsically coupled with each other. In stress-induced cyclic phase transition, mechanical and thermal aspects of the material should be considered in parallel.

The second challenging issue is the instabilities of dynamic system with NiTi SMA, such as chaos, bifurcation and cusp catastrophe (Nayfeh and Mook, 2008; Schmidt and Tondl., 1986; Stoker, 1966), due to the intrinsic nonlinear and hysteretic behavior of the material in stress-induced phase transition. A typical feature of cusp catastrophe is the jump phenomenon when a nonlinear oscillator is subjected to an external periodic excitation as shown in Fig. 2. When increasing the excitation frequency or amplitude, the dynamic response ascends on the Frequency Response Curve (FRC) or Amplitude Response Curve (ARC) along A-B-C and then jumps up from C to D. However, when decreasing the frequency or amplitude, the dynamic response takes a different routine and goes by E-D-F and then suddenly jumps down from F to B. Between the jump-up and jump-down, a multi-valued metastable region (B-C and D-F) exists and the dynamic response may overcome the stability barrier and switch from one branch of solution to the alternative metastable branch, giving rise to the jump phenomena (Scheffer et al., 2001; Scheffer and Carpenter, 2003). Specifically, for a vibration system with softening nonlinear NiTi SMA, multiple metastable states of the dynamic response are inevitable and the jumps between these metastable states may occur when external and internal control parameters fall into certain regions. These regions must be well identified and excluded in the applications.

Last but not least, direct experimental observation and systematic investigation on the jump phenomena in the thermomechanical responses of nonlinear dynamic system with NiTi SMA require careful design of the testing platform and precise acquisition of large quantity of experimental data. In modeling and simulation, there has been a lot of work over the past two decades. The softening nonlinear features and jumps of thermomechanical responses (Bernardini and Rega, 2005; Lagoudas et al., 2005; Machado and Lagoudas, 2006; Masuda and Noori, 2002; Piccirillo et al., 2010a, 2010b; Seelecke, 2002), the softening-hardening nonlinear features (Bernardini and Vestrioni, 2003; Lacarbonara et al., 2004; Masuda and Noori, 2002) and effects of system parameters on the jump phenomenon (Bernardini and Rega, 2010; Bernardini and Vestrioni, 2003; Lacarbonara et al., 2004; Lagoudas et al., 2005; Machado and Lagoudas, 2006; Masuda and Noori, 2002; Seelecke, 2002) have been well identified. In addition to jump phenomenon, chaos and bifurcation (Bernardini and Rega, 2005; Dos Santos and Savi, 2009; Lacarbonara et al., 2004; Machado et al., 2007a, 2007b, 2003, 2004; Piccirillo et al., 2008, 2009a, 2009b; Savi, 2015; Savi and Nogueira, 2010; Savi and Pacheco, 2002; Savi et al., 2002) were also studied in softening nonlinear dynamic systems with NiTi SMA. In sharp contrast, much less has been achieved on the experimental aspect. Some early experimental works (Collet et al., 2001; Feng and Li, 1996; Graesser, 1995; Lammering and Schmidt, 2001; Li and Feng, 1997; Schmidt and Lammering, 2004a, 2004b) have indentified the softening nonlinear features on the transmissibility curve. The resonance frequencies and peak responses strongly depend on the external excitations. The jump of rotation on the FRC in torsional vibration of NiTi wire was directly identified for the first time in 2012.
(Doaré et al., 2012; Moussa et al., 2012). The simultaneous jumps of rotation and temperature were further quantified for a mass-NiTi wire torsional vibration system in the frequency domain (Xia and Sun, 2015). However, the existing works are limited to the frequency domain only and there has been no report on the thermomechanical responses in the amplitude and time domain. In particular, there is a lack of systematic experimental investigation on the jumps in the frequency, amplitude and time domains, which impedes a comprehensive understanding of the nonlinear phenomenon and also prevents the NiTi SMA from further applications.

This paper aims at identifying the alternative stable states and the jump phenomena of dynamic responses in the frequency, amplitude and time domains for a nonlinear torsional vibration system with NiTi wire. Basic thermomechanical properties of the material and the testing platform are described in Section 2. The alternative stable states and the jumps of thermomechanical responses in the frequency, amplitude and time domains are reported and discussed in Section 3. The summary and key conclusions are given in Section 4.

2. Material, sample preparation and experimental setup

A polycrystalline superelastic Ni-49.6 at.% Ti SMA wire (diameter \( d = 1.63 \text{ mm} \), total length \( L_{\text{total}} = 60.00 \text{ mm} \) and gauge length \( L = 20.00 \text{ mm} \) from Nitinol, USA) is used as a softening nonlinear damping spring in the torsional vibration system in Fig. 3. Basic mechanical and thermal properties of the NiTi wire are listed in Table 1. Room temperature (\( T_0 \)) is maintained at \( T_0 \approx 25 °C \) during the experiments and therefore superelasticity of the specimen can be guaranteed. As a long cylindrical shaft under torsion, the NiTi wire will first experience martensitic phase transition on the cylindrical surface at the rotation of \( \varphi \approx \frac{2\pi}{3} = 154.7° \) (\( \gamma \) is the shear strain at the start of phase transition). Using the shear modulus of austenite phase (\( G = 16.5 \text{ GPa} \)) and rotational momentum of
inertia \((J = 0.032 \text{ kg} \cdot \text{m}^2)\) of the torsional vibration system, the primary resonance frequency in the linear elastic region is estimated as \(f_0 = \frac{1}{2\pi} \sqrt{\frac{\pi G d L}{J}} \approx 0.673 \text{ Hz}\).

Before the torsional vibration tests, the NiTi wire was trained by two-way cyclic torsion under quasi-isothermal conditions \((f = 0.001 \text{ Hz})\) for 200 cycles in Fig. 4(a) to ensure a repeatable torque-angle relation. The measured torque-angle relations \((M = M(\varphi, f))\) at 1st, 50th, 100th, 150th and 200th cycle are examined in particular. It is seen that the \(M \sim \varphi\) relation become completely repeatable after 150th cycle (in red). After training, \(M \sim \varphi\) relation of the NiTi wire was characterized at \(\varphi = \pm 100^\circ\) and different frequencies of \(f = 0.001/0.35/0.45/0.55/0.65/0.75 \text{ Hz}\). The steady state responses are summarized in Fig. 4(b) and it is seen that the non-isothermal \(M \sim \varphi\) relations significantly deviate from the isothermal case \((f = 0.001 \text{ Hz} \text{ denoted by dashed line})\) in pure torsion. However, in the frequency range of \(f \in [0.35 \text{ Hz}, 0.75 \text{ Hz}]\), which is the focus of this paper, the differences among these steady state non-isothermal \(M \sim \varphi\) relations are negligible and the \(M \sim \varphi\) relations can be represented by the same curve.

At an intermediate frequency of \(f = 0.60 \text{ Hz}\), pure torsion tests are performed at \(\varphi = \pm 10^\circ/\pm 20^\circ/\pm 30^\circ/\pm 40^\circ/\pm 50^\circ/\pm 60^\circ/\pm 70^\circ/\pm 80^\circ/\pm 90^\circ/\pm 100^\circ\) and all of the steady state \(M \sim \varphi\) relations collapse on a single curve in Fig. 4(b). It is noticed that due to the internal elastic core inside the NiTi wire during torsion, the rotational angle cannot be totally recovered at \(M = 0\) and there is a very small residual rotation \((1^\circ \sim 2^\circ)\). In order to check if there is any cyclic plasticity of the NiTi wire during the vibration tests, each specimen was examined after the dynamic tests by pure torsion at \(f = 0.60 \text{ Hz}\) and \(\varphi = \pm 100^\circ\). It’s seen from Fig. 4(c) that degradation of the NiTi wires during the torsional vibration tests is indeed ignorable.

As shown in Fig. 3, the dynamic testing platform consists of the following four parts. (1) An electric rotational stepping motor on an electromagnetic test instrument (ElectroForce 3510 from BOSE, USA) to provide external sinusoidal angular excitation \((\alpha = \text{Asin}(2\pi ft))\). (2) NiTi wire and clamping with special designed teeth to avoid sliding. (3) A beam with two mass blocks attached at

![Dynamic response of vibration system in the frequency and amplitude domains](image1)

![Jump phenomenon of a softening nonlinear vibration system](image2)

**Fig. 2.** (a) FRC and ARC of a softening/linear/hardening vibration system; (b) jump phenomenon of a softening nonlinear vibration system in the frequency and amplitude domains.
its two ends to provide rotational momentum of inertia \( J = 0.032 \, \text{kg} \cdot \text{m}^2 \) at the upper end of the NiTi wire. The beam is held by a single direction thrust ball bearing so as to rotate freely without axial force. (4) Synchronized data acquisition of the mechanical and thermal responses. Relative rotational angle of the NiTi wire \( (\theta_t - \beta_t - \alpha_t) \) is obtained from the external excitation \( \alpha_t \) and rotation of the beam \( \beta_t \) with a rotary motion sensor (PS-2120A from PASCO, USA). Volume average temperature of the NiTi wire \( \left( \int T_x \, dx / V \right) \), as the thermal response of the dynamic system, is measured by an infrared camera (SC7700M from FLIR, USA). The NiTi wire is painted with matt black paint (TS-6 from Tamiya, Japan) to enhance the emissivity for thermal imaging. (Remark:

**Table 1**

Mechanical and thermal properties of the NiTi wire.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Austenite finish temperature</td>
<td>( A_f ) \text{ °C}</td>
</tr>
<tr>
<td>Shear modulus in austenite phase</td>
<td>( G ) \text{ GPa}</td>
</tr>
<tr>
<td>Shear strain at the start of phase transition</td>
<td>( \gamma_s ) \text{ %}</td>
</tr>
<tr>
<td>Maximum phase transition shear strain</td>
<td>( \gamma_{PT} )</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>( k ) \text{ W/K-m}</td>
</tr>
<tr>
<td>Specific latent heat</td>
<td>( l_0 ) \text{ J/m}^3</td>
</tr>
<tr>
<td>Specific heat capacity</td>
<td>( \lambda = \rho c_p ) \text{ J/m}^3 K</td>
</tr>
<tr>
<td></td>
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</tr>
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**Fig. 3.** Experimental setup and synchronized measurement of thermomechanical responses in the nonlinear torsional vibration system.
in the testing frequency range of \( f \in [0.4 \text{ Hz}, 0.7 \text{ Hz}] \), the characteristic heat conduction time \( t_\lambda = \frac{d^2}{4k} \approx 0.011 \text{ s} \) over the circular cross-section of the NiTi wire is much shorter than the characteristic deformation time \( t_p = \frac{1}{f} \in [1.43 \text{ s}, 2.50 \text{ s}] \). So the temperature field over the cross-section can be regarded as uniform and is equal to the temperature at the surface).

To validate the dynamic testing platform, torsional vibration at the excitation of \( A = 12^\circ \) and \( f = 0.580 \text{ Hz} \) is taken as an example for analysis. From the simultaneously monitored axial displacement and axial force in Fig. 5(a), it is seen that the maximum compressive force on the specimen is \(-4 \text{ N}\) (equivalent to a compressive stress of \(1.92 \text{ MPa}\)), far below the elastic buckling load of \(-63.24 \text{ N}\) (Timoshenko and Goodier, 1969). The maximum tensile stress is \(0.48 \text{ MPa}\) and maximum tensile strain is \(0.15\%\). The axial stress and axial strain are negligible compared with the applied shear stress and shear strain. Consequently, torsional vibration tests with ignorable axial stress and axial strain can be guaranteed. In the evolution of thermomechanical responses in Fig. 5(b), both
the rotational angle ($\theta(t)$) and temperature ($T(t)$) experience a transient state and a steady state under external excitation and a free damping state after the excitation is off. $\theta(t)$ oscillates symmetrically around 0° but $T(t)$ is non-symmetric since the mean temperature increases in the transient state due to the accumulation of hysteresis heat and heat exchange with the ambient (Yin et al., 2014). Fig. 5(c) shows the synchronized measurements of $\alpha_t$, $\beta_t$, $\theta_t$ and $T_t$ in the transient state and (d) oscillation of $\theta(t)$ and $T(t)$ in the steady state.

Fig. 5. (a) Axial force and axial displacement of the NiTi wire during torsional vibration test; (b) evolutions of $\theta(t)$ and $T(t)$ in the time domain; (c) synchronization of $\alpha_t$, $\beta_t$, $\theta_t$ and $T(t)$ in the transient state and (d) oscillation of $\theta(t)$ and $T(t)$ in the steady state.
3. Experimental results and discussions

3.1. Alternative stable states and their jumps in the frequency domain

Under a given external excitation amplitude and frequency, the steady state peak-to-valley magnitudes of rotational angle and temperature \( \Delta \theta = \theta_{\text{max}} - \theta_{\text{min}} \) and \( \Delta T = T_{\text{max}} - T_{\text{min}} \) in one cycle are quantified as the mechanical and thermal responses of the nonlinear vibration system. To capture the alternative stable states and the consequent jumps of \( \Delta \theta \) and \( \Delta T \) in the frequency domain under given excitation amplitude \( A \), the applied excitation frequency \( f \) should well span the primary resonance frequency \( f_0 \). In the experiment, for each of the five excitation amplitudes \( (A = 1.5^\circ, 3^\circ, 6^\circ, 9^\circ \text{ or } 12^\circ) \), a total of 61 different excitation frequencies are selected in the range of \( f \in [0.400 \text{ Hz}, 0.700 \text{ Hz}] \) around \( f_0 \). Independent experiment is performed at each frequency according to ascending order (from 0.400 Hz up to 0.700 Hz at 0.005 Hz interval) to search for the jump-up phenomena and descending order (from 0.700 Hz down to 0.400 Hz at 0.005 Hz interval) to search for the jump-down phenomena respectively. The obtained FRCs of \( \Delta \theta \) and \( \Delta T \) are summarized on the left side of Fig. 6.

It is seen that under small excitation amplitude (e.g., \( A = 1.5^\circ \) in black), the deformation of NiTi wire is still in the linear elastic range and \( \Delta T \) is negligible. As the excitation amplitude gradually increases (e.g., \( A = 3^\circ \) in red), \( \Delta \theta \) and \( \Delta T \) become larger as the NiTi wire enters the phase transition region and significant asymmetries appear in the FRCs. At the frequency of 0.625 Hz, the FRCs become steep, implying the jump and drop in the FRCs of \( \Delta \theta \) and \( \Delta T \). The difference of the FRCs following ascending and descending frequency orders becomes distinct. At larger excitation amplitude (e.g., \( A = 6^\circ \) in green), the jump-up of \( \Delta \theta \) following ascending frequency order and jump-down of \( \Delta \theta \) following descending frequency order are eventually captured at \( f_{\text{up}} = 0.580 \text{ Hz} \) and \( f_{\text{down}} = 0.570 \text{ Hz} \) respectively in the experiments. Simultaneous jump-up and jump-down of \( \Delta T \) at exactly the same \( f_{\text{up}} \) and \( f_{\text{down}} \) are observed on the FRC of temperature response in Fig. 6(b). When further increasing the excitation amplitudes (e.g., \( A = 9^\circ \) or \( 12^\circ \) in blue or cyan), the jumps of \( \Delta \theta \) and \( \Delta T \) in the frequency domain become more significant.

The jump frequencies \( (f_{\text{up}} \text{ and } f_{\text{down}}) \), jump frequency gap \( (\Delta f = f_{\text{up}} - f_{\text{down}}) \) and jump magnitudes of thermomechanical response \( (\Delta \theta_{\text{up}}, \Delta \theta_{\text{down}}, \Delta T_{\text{up}}, \text{ and } \Delta T_{\text{down}}) \) are summarized in Fig. 7. It is seen that both \( f_{\text{up}} \) and \( f_{\text{down}} \) are very close to but smaller than the estimated primary resonance frequency \( (f_0 \approx 0.673 \text{ Hz}) \). As the excitation amplitude \( A \) increases, both \( f_{\text{up}} \) and \( f_{\text{down}} \) decrease but the frequency gap \( \Delta f \) increases. At the same time, the jump magnitudes of thermomechanical responses \( (\Delta \theta_{\text{up}}, \Delta \theta_{\text{down}}, \Delta T_{\text{up}}, \text{ and } \Delta T_{\text{down}}) \) keep increasing with \( A \).

Physically phase transition of NiTi SMA is the origin of softening nonlinearity and thermomechanical coupling in the behavior of the material. Thermomechanical coupling effects in the dynamic responses are indeed very complicated and mathematically demanding in modeling and analysis. Therefore, as a first approximation in this paper, the mechanical response and thermal

![Fig. 6.](image-url) (a) Comparison of the steady state \( \Delta \theta \) between the experimental data and theoretical modeling under the excitation of \( A = 1.5^\circ/3^\circ/6^\circ/9^\circ/12^\circ \); (b) comparison of the steady state \( \Delta T \) between the experimental data and theoretical modeling under the excitation of \( A = 1.5^\circ/3^\circ/6^\circ/9^\circ/12^\circ \). (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)
response are treated as one-way weak coupling: thermal response is determined by mechanical response and there is no vice versa. Actually, the temperature oscillation will in turn affect the rotational angle in the vibration system (Xia and Sun, 2015) but it's not considered in this paper.

For the purpose of simplicity in the interpretation of jump phenomena, the softening nonlinear hysteretic torque-angle relation of NiTi wire during phase transition is treated as an effective nonlinear spring with equivalent viscous damping so as to have an analytical form of the dynamic responses. With above simplification and approximation, an effective Duffing oscillation model and a lumped heat transfer analysis (Xia and Sun, 2015) are proposed to quantify the mechanical and thermal responses of the nonlinear torsional vibration system with NiTi wire. The phase transition nonlinearity and dissipative property of the NiTi wire in torsion are separately modelled as an equivalent cubic nonlinear spring \( (M_k \theta \dot{\theta}^3) \) and an equivalent angular viscous damper \( (M_c \dot{\theta}) \) as shown in Fig. 8(a).

The equation of motion in Fig. 3 can be expressed as,

\[
J \ddot{\theta} = -c_{\text{eq}} \dot{\theta} - k_\theta \theta - k_\theta \dot{\theta}^3
\]

Replacing \( \beta \) by \( \theta \) and \( \alpha (\beta = \theta + \alpha = \theta + \text{Asin}(2\pi f t)) \), we have

\[
J \ddot{\theta} + c_{\text{eq}} \dot{\theta} + k_\theta \theta + k_\theta \dot{\theta}^3 = 4AJ^2 \pi f^2 \sin 2\pi ft
\]

Assuming that the steady state solution is a sinusoidal periodic function with frequency \( f \) at phase angle \( \chi \),

\[
\theta = \frac{\Delta \theta}{2} \cos(2\pi ft - \chi)
\]

The steady state response is obtained with the method of multiple scales.

Fig. 7. (a) Experimental data and (b) theoretical modeling results of the jump frequencies \( f_{\text{up}} \) and \( f_{\text{down}} \), jump frequency gap \( (\Delta f) \), jump magnitudes of rotational angle \( (\Delta \theta_{\text{up}} \text{ and } \Delta \theta_{\text{down}}) \) and jump magnitudes of temperature \( (\Delta T_{\text{up}} \text{ and } \Delta T_{\text{down}}) \) at different excitation amplitudes.
Meanwhile, the “backbone” curve in the frequency domain is given by

$\Delta \theta = 4 \sqrt{\frac{4\pi^2 f^2 J - k_1}{3k_2}}$

(3)

Assuming the shear behavior of NiTi wire under partial phase transition in torsion as ideal “elastoplastic” type with perfect plasticity in Fig. 8(b), a lumped heat transfer analysis is performed to establish the relationship between the temperature oscillation ($\Delta T$) and the rotation ($\Delta \theta$) of the NiTi wire,

$\Delta T = \begin{cases} 0 & \left(\frac{\Delta \theta \leq 4L\gamma}{d}\right) \\ \frac{d}{\Delta \theta}\Delta \theta + \frac{16L\gamma^2}{3\pi^2 d^2} - \frac{\gamma}{\eta\gamma' \Delta \theta} & \left(\frac{\Delta \theta > 4L\gamma}{d}\right) \end{cases}$

(4)

In summary, Eq. (1) is the governing equation of the nonlinear dynamic system under external sinusoidal excitation. Eq. (2) is an implicit solution of rotational angle amplitude ($\Delta \theta$) as a function of the external excitation ($f$ and $A$). Eq. (3) determines the “backbone” curve of FRCs in the frequency domain. Eq. (4) gives the temperature oscillation ($\Delta T$) at a given rotational angle ($\Delta \theta$). Eqs. (2) and (4) give the approximate solutions of the thermomechanical responses in both frequency and amplitude domains.

The $c_{eq}$ is obtained by equaling the steady state hysteresis loop area of the material in cyclic pure torsion to the energy dissipation of a viscous damper in a mechanical vibration system. At different rotational frequency and angle, the $c_{eq}$ is almost the same and
therefore taken as a constant in the following theoretical analysis. The $k_1$ and $k_2$ come from the cubic nonlinear fitting of the steady state torque angle relation of the NiTi wire. Therefore, the effective damping coefficient and equivalent spring constants are independently quantified as: $c_{eq} = 0.0052 \text{ N-m-s}$, $k_1 = 0.480 \text{ N-m}$ and $k_2 = -0.105 \text{ N-m}$. Together with the basic material properties in Table 1, the FRCs of the mechanical and thermal responses are obtained from the above simplified theoretical analysis and plotted on the right side of Figs. 6 and 7 for comparison.

It is seen that the FRCs strongly bend to the left side, showing typical softening nonlinear properties. As the excitation amplitude $A$ increases, the magnitudes of thermomechanical response grow larger and the jump frequencies ($f_{\text{up}}$ and $f_{\text{down}}$) consistently shift to the left. At the same time, the jump frequency gap ($\Delta f$) and jump magnitudes ($\Delta \theta_{\text{up}}$ and $\Delta T_{\text{up}}$, $\Delta \theta_{\text{down}}$ and $\Delta T_{\text{down}}$) keep increasing with the excitation amplitude. We see that the general features of the FRCs from the experiments and modeling agree with each other. Although there are some quantitative errors, the thermomechanical responses of a nonlinear torsional vibration system with NiTi wire in the frequency domains can be well quantified by the simplified Duffing oscillation model and lumped heat analysis.

3.2. Alternative stable states and their jumps in the amplitude domain

In the amplitude domain, to search for the alternative stable states and capture their jumps, the excitation frequency $f$ is kept as a constant while the excitation amplitude $A$ is slowly changed following ascending and descending orders respectively. In the experiments, five excitation frequencies ($f = 0.50 \text{ Hz}$, $0.52 \text{ Hz}$, $0.54 \text{ Hz}$, $0.56 \text{ Hz}$ and $0.58 \text{ Hz}$) are selected to perform the torsional vibration tests. At each frequency, $A$ is taken from the range of $A \in [6.0^\circ, 12.0^\circ]$ and a total of 122 independent experiments are performed from $6.0^\circ$ up to $12.0^\circ$ and from $12.0^\circ$ down to $6.0^\circ$ with $0.1^\circ$ interval. The steady state $\Delta \theta$ and $\Delta T$ are measured and plotted against $A$ as ARCs on the left side of Fig. 9.

Different from the FRCs in Section 3.1, the ARCs do not have a peak response in the intermediate region and both $\Delta \theta$ and $\Delta T$ monotonically increase with $A$ at each excitation frequency. At a small excitation frequency (e.g., $f = 0.50 \text{ Hz}$ in black), the deformation of NiTi wire is within the linear elastic region of the material. No jump phenomena are observed in the testing amplitude range. When $f$ is increased up to $0.52 \text{ Hz}$ in red, simultaneous jump-up of $\Delta \theta$ and $\Delta T$ are observed on the ARCs at $A_{\text{up}} = 11.5^\circ$ following ascending order and simultaneous jump-down occurs at $A_{\text{down}} = 10.2^\circ$ following descending order, thus bringing a multi-valued region ($\Delta A = A_{\text{up}} - A_{\text{down}}$) in the amplitude domain. As $f$ is further increased to $0.54 \text{ Hz}$ and $0.56 \text{ Hz}$ (in green and blue respectively), the lower branch will drift up while the upper branch will drift down. Consequently the jump magnitudes of both rotational angle ($\Delta \theta_{\text{up}}$ and $\Delta \theta_{\text{down}}$) and temperature oscillation ($\Delta T_{\text{up}}$ and $\Delta T_{\text{down}}$) will decrease with $f$. At the same time, the jump amplitudes ($A_{\text{up}}$ and $A_{\text{down}}$) shift to the left with the increase of excitation frequency. At very large frequency (e.g., $f = 0.58 \text{ Hz}$ in cyan), the entire ARCs are on the upper branch without jump phenomena in the testing amplitude range, though they may exist at smaller amplitude.

(a) Experiments and modeling of the ARCs of rotation

(b) Experiments and modeling of the ARCs of temperature

Fig. 9. (a) Comparison of the steady state $\Delta \theta$ between the experimental data and theoretical modeling under the excitation of $f = 0.50 \text{ Hz}/0.52 \text{ Hz}/0.54 \text{ Hz}/0.56 \text{ Hz}/0.58 \text{ Hz}$; (b) comparison of the steady state $\Delta T$ between the experimental data and theoretical modeling under the excitation of $f = 0.50 \text{ Hz}/0.52 \text{ Hz}/0.54 \text{ Hz}/0.56 \text{ Hz}/0.58 \text{ Hz}$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)
To quantify the ARCs of thermomechanical responses in the amplitude domain, Eqs. (2) and (4) are used with the same parameters as Section 3.1 and the theoretical modeling results are shown on the right side of Figs. 9 and 10. It is seen that the nonlinear features of experimental data are well predicted by the theoretical modeling. Jump-up and jump-down are captured on the ARCs following ascending and descending amplitude paths in the amplitude domain. The modeling results also confirm that the jump amplitudes ($A_{up}$ and $A_{down}$), jump amplitude gap ($\Delta A$), jump magnitudes of rotational angle ($\Delta \theta_{up}$ and $\Delta \theta_{down}$) and jump magnitudes of temperature ($\Delta T_{up}$ and $\Delta T_{down}$) decrease with the excitation frequency ($f$) in the testing range.

3.3. Alternative stable states and their jumps in the time domain

In the frequency and amplitude domains, the existence of alternative stable states only manifests as separated branches of solutions and the consequent jumps between them on the FRCs and ARCs. To explore how the transition or jump phenomenon takes place, we have to move to the time domain. The multi-valued region between jump-down and jump-up in the frequency and amplitude domains are actually metastable and prone to change when subject to disturbance. When the external excitation frequency or amplitude falls into such metastable region, depending on the magnitude and direction of disturbance, the metastable state of the dynamic responses may quickly shift to an alternative one in the time domain, illustrating the instability nature of the nonlinear vibration system.

In this section, the torsional vibration tests are performed to capture the jump phenomena in the time domain. To search for the critical external excitation, the excitation amplitude is gradually reduced to narrow the frequency gap ($\Delta f$) between jump-up and jump-down in the frequency domain (see Section 3.1). When $A = 3^\circ$, the jump-up and jump-down almost overlap ($\Delta f \approx 0$) and then the excitation frequency is changed very slowly around the jump frequency to further identify the critical excitation frequency.

Fig. 10. (a) Experimental data and (b) theoretical modeling results of the jump amplitudes ($A_{up}$ and $A_{down}$), jump amplitude gap ($\Delta A$), jump magnitudes of rotational angle ($\Delta \theta_{up}$ and $\Delta \theta_{down}$) and jump magnitudes of temperature ($\Delta T_{up}$ and $\Delta T_{down}$) at different excitation frequencies.
Under the excitation of $A = 3^\circ$ and $f = 0.62160$ Hz, the evolutions of $\theta(t)$ and $T(t)$ are plotted in Fig. 11. It is seen at $f = 0.62160$ Hz the small steady state $\Delta \theta$ and $\Delta T$ are typical of the dynamic responses on the lower branch of FRC (or ARC) when $f < f_{\text{down}}$ (or $A < A_{\text{down}}$). On the other hand, at $f = 0.62180$ Hz in Fig. 11(b), the system response reaches a different steady state with large oscillation magnitude, which is typical of the response on the upper branch when $f > f_{\text{up}}$ (or $A > A_{\text{up}}$). Different from the
below two cases, at the critical excitation of $A = 3^\circ$ and $f_0 = 0.62170 \text{ Hz}$, evolutions of the thermomechanical responses experience a spontaneous transition from the lower branch to the alternative upper branch in the time domain. After the initial transient state, the dynamic responses first settle down to the metastable state I, very similar to the lower branch in Fig. 11(a). After 150 s, the dynamic response gradually but spontaneously shifts to a metastable state II, which is close to the upper branch response with large oscillation magnitude in Fig. 11(b). In this way, we captured the spontaneous jump phenomena of thermomechanical responses from the lower branch of solution to the alternative upper branch in the time domain in one single experiment. When further enlarging and examining $\theta(t)$ and $T(t)$ of the two metastable states in Fig. 12, the phase angle change becomes very distinct. In metastable state I, rotation of the specimen is phase lagging referring to the external excitation. However, in metastable state II, the mechanical response is leading the phase. In the new transient state between these two metastable states, the thermomechanical responses experience phase inversion and this is a key characteristic of the jump phenomena.

The above spontaneous jump phenomena from the lower branch to the alternative upper branch are due to the internal intrinsic disturbances (e.g., friction, temperature fluctuation, etc.). The influence of excitation and disturbance on the jump phenomena in the time domain will be further investigated by introducing the external disturbance by hand in the following. Positive disturbance is defined as pushing forward the mass blocks to enlarge the rotational angle while negative disturbance pulls them back to reduce the rotation during the steady state oscillation.

At $A = 9^\circ$, torsional vibration tests are performed at $f = 0.450 \text{ Hz}$, $0.600 \text{ Hz}$ and $0.531 \text{ Hz}$, corresponding to the stable lower branch, stable upper branch and metastable region respectively in the frequency domain in Fig. 6. At $f = 0.450 \text{ Hz}$ and $0.600 \text{ Hz}$, a negative and a positive disturbance are exerted at about 100 s and 200 s respectively and the thermomechanical responses are shown in Fig. 13(a) and (b). It is seen that at these two frequencies away from the metastable region in the frequency domain, neither a positive nor negative disturbance can induce the jump phenomena in the time domain. This is because at excitations in the stable region, only one stable focus exists and therefore there is always only one stable solution. At $f = 0.531 \text{ Hz}$ within the metastable region, a positive and a negative disturbance are applied at about 100 s in the steady state in two independent experiments in Fig. 13(c). It is seen that the new transient state induced by the negative disturbance leads to the jump phenomena while the new transient state after a positive disturbance quickly dies away without jumps. Such strong direction dependence of the jump phenomena can be qualitatively interpreted by state plane as shown on the right side of Fig. 13. Different external disturbance can drag the steady state response towards different direction. A negative disturbance tends to move the dynamic response towards the upper branch stable focus and even go across the boundary if the disturbance is large enough, thus bringing about a state shift from the lower branch of solution to the alternative upper branch. On the other hand, a positive disturbance drags the steady state response towards the lower branch stable focus further away from the saddle point, therefore always staying on the lower branch without jumps to the alternative upper branch in the time domain.

Similar influences of external excitation on the jump phenomena are also observed when the excitation is within the metastable region in the amplitude domain at $f = 0.56 \text{ Hz}$ in Fig. 9. The dynamic responses are stable at $A = 6.0^\circ$ on the lower branch and $A = 8.0^\circ$ on the upper branch but metastable at $A = 6.6^\circ$ in the amplitude domain. At $A = 6.0^\circ$ and $8.0^\circ$, there are always no jumps in the time domain regardless of external disturbances in Fig. 14(a) and (b). At $A = 6.6^\circ$, a negative disturbance can induce the jump phenomena in Fig. 14(c). However, the jumps do not occur after a positive disturbance at the same external excitation. It is seen that the jump phenomena in the time domain are significantly affected by the excitation and external disturbance. When the external excitation is in the stable (or single-valued) region, there are always no jump phenomena regardless of external disturbance. At excitations within the metastable (or multi-valued) region, a positive disturbance cannot induce such jumps as well. Only a negative external disturbance under the external excitations within the metastable region in the frequency or amplitude domain can trigger the jump phenomena of thermomechanical responses in the time domain.

To further investigate the effects of external disturbance on the jumps, nonlinear dynamic tests are performed in the metastable region with small and large external disturbances. At $A = 9^\circ$ and $f = 0.530 \text{ Hz}$, the system is metastable in the frequency domain in Fig. 6. With small negative disturbance at 100 s and large negative disturbance at 200 s in the steady state, evolutions of the

![Fig. 12. Comparison of the oscillations of $\theta(t)$ and $T(t)$ between metastable state I and metastable state II.](image-url)
(a) $A = 9.0^\circ$ and $f = 0.450$ Hz (stable lower branch in the frequency domain)

(b) $A = 9.0^\circ$ and $f = 0.600$ Hz (stable upper branch in the frequency domain)

(c) $A = 9.0^\circ$ and $f = 0.531$ Hz (metastable region in the frequency domain)

Fig. 13. Evolutions of $\theta(t)$ and $T(t)$ at the excitation of $A = 9.0^\circ$ and (a) $f = 0.450$ Hz, (b) $f = 0.600$ Hz, (c) $f = 0.531$ Hz with external negative and positive disturbances.
thermomechanical responses are shown in Fig. 15(a). It is seen that the new transient state induced by small disturbance dies away after some time while the large negative disturbance triggers the jump phenomena in the time domain. This may be because the large negative disturbance provides enough energy to overcome the barrier and go across the boundary to reach the upper branch stable.

(a) $A = 6.0^\circ$ and $f = 0.56$ Hz (stable lower branch in the amplitude domain)

(b) $A = 8.0^\circ$ and $f = 0.56$ Hz (stable upper branch in the amplitude domain)

(c) $A = 6.6^\circ$ and $f = 0.56$ Hz (metastable region in the amplitude domain)

Fig. 14. Evolutions of $\theta(t)$ and $T(t)$ at the excitation of $f = 0.56$ Hz and (a) $A = 6.0^\circ$, (b) $A = 8.0^\circ$, (c) $A = 6.6^\circ$ with external negative and positive disturbances.

It is seen that the new transient state induced by small disturbance dies away after some time while the large negative disturbance triggers the jump phenomena in the time domain. This may be because the large negative disturbance provides enough energy to overcome the barrier and go across the boundary to reach the upper branch stable.
focus in the state plane. If the exerted disturbances are positive as shown in Fig. 15(b), neither a small nor a large disturbance can induce the jump phenomena and this is consistent with the previous results. At the excitation of $A = 9.2^\circ$ and $f = 0.54$ Hz, within the metastable region in the amplitude domain in Fig. 9, similar results are also observed as shown in Fig. 16. The large negative disturbance can induce the jump phenomena in the time domain while a small negative disturbance and positive disturbance (no matter small or large) can only lead to a new transient state which dies away quickly without jump phenomena in the time domain. It is seen that the jumps are also inherently associated with the disturbance: only a proper external disturbance (both in direction and magnitude) can trigger the jump phenomena of thermomechanical responses in the time domain.

The multi-values of FRC and ARC or alternative stable states of the nonlinear dynamic system as well as their jumps are summarized in terms of stability landscape in Fig. 17, where stability of the dynamic system in steady state depends on both excitation and response. When the excitations fall into the stable region, the dynamic response is single-valued, corresponding to only one minimum on the stability curve and there are always no jump phenomena regardless of disturbance. When the excitation is within the metastable region, there are two minima on the stability curve, implying two metastable states on the lower and upper branches respectively. The external disturbance can help the system to overcome the energy barrier to reach a more favorable alternative metastable state, bringing about the jump phenomena in multiple domains. At the critical conditions, the jump phenomena will occur spontaneously without observable external disturbance.

Prior to this paper, most of the research focused on the theoretical modeling and numerical simulation of the jump phenomena in a nonlinear vibration system. The very limited experimental work was mainly on the mechanical responses in the frequency domain. In this paper, both mechanical response (i.e., rotation) and thermal response (i.e., temperature) of nonlinear torsional vibration system with NiTi SMA are investigated and high-resolution synchronized data of thermomechanical responses are obtained. In addition, the experimental exploration of the jump phenomena is further expanded from the frequency domain to the amplitude and time domains to get a more comprehensive understanding of the instability phenomena in the nonlinear dynamic system with NiTi SMA. In particular, for the first time, the transition process from one branch of solution to the alternative stable branch is captured in the time domain by experiment with and without external disturbance. The instability and consequent transition process are qualitatively interpreted by the stability landscape from the energy point of view.

4. Summary and conclusions

In this paper, experimental investigations of the dynamic responses of a nonlinear torsional vibration system with phase transformable NiTi SMA are performed for the first time in the frequency, amplitude and time domains. Alternative stable states and...
(a) Negative disturbances at $A = 9.2^\circ$ and $f = 0.54$ Hz (metastable in the amplitude domain)

(b) Positive disturbances at $A = 9.2^\circ$ and $f = 0.54$ Hz (metastable in the amplitude domain)

Fig. 16. Evolutions of $\theta(t)$ and $T(t)$ at the excitation of $f = 0.54$ Hz and $A = 9.2^\circ$ (a) with small and large negative disturbances, (b) with small and large positive disturbances.

Fig. 17. Stability landscape of the steady state dynamic response.
simultaneous jump phenomena of thermomechanical responses are quantified on both FRCs and ARCs. The fast transition process between the two metastable states is captured and the effects of disturbance on triggering the jumps are examined in the time domain. Systematic data of the metastable region where jump phenomena may happen are acquired by the experiments. The main conclusions of this paper are the followings:

1. The nonlinearity and damping capacity of the NiTi SMA during phase transition play a critical role on the stability of the nonlinear dynamic responses. By changing the amount of phase transition of the NiTi wire, the external excitation amplitude A can significantly influence the stability of dynamic responses. As A increases, both lower and upper branches of the FRCs drift up and $f_{up}$ and $f_{down}$ decrease, but $\Delta f$, $\Delta f_{up}$, $\Delta f_{down}$, $\Delta T_{up}$, $\Delta T_{down}$ increase.

2. Simultaneous jump phenomena of mechanical response and thermal response exist on the ARC in the amplitude domain for nonlinear vibration with NiTi SMA. As excitation frequency f increases, the lower branch of the ARCs will drift up while the upper branch will drift down. The $A_{up}$, $A_{down}$, $\Delta A$ and $\Delta \theta_{up}$, $\Delta \theta_{down}$, $\Delta \theta_{up}$, $\Delta \theta_{down}$ keep decreasing with f.

3. Jump phenomena between the alternative stable states in the time domain are inherently associated with disturbance. They can either occur spontaneously at critical excitations without observable external interference or occur under external disturbance of appropriate direction and magnitude. Based on the experiments in the three domains, alternative stable states can be accurately identified and the stability landscape is constructed to interpret the occurrence of jump phenomena and to manage the vibration system. Examples include promoting the desired dynamic responses and preventing the undesired ones.

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