On non-monotonic rate dependence of stress hysteresis of superelastic shape memory alloy bars

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\begin{abstract}
This paper presents a simple thermo-mechanical model to explain and quantify the observed strain-rate dependence of the stress hysteresis of shape memory alloys (SMAs) bars/strips during stress-induced forward/reverse phase transition with latent heat release/absorption. By solving the convective heat transfer equation and employing the temperature dependence of the SMA’s transformation stresses, we are able to prove that the stress hysteresis depends non-monotonically on the applied strain rate with a peak appearing at an intermediate strain rate. We further showed that such a non-monotonic rate dependence is governed by the competition of phase-transition time (or latent-heat release/absorption time) and the time of heat exchange with the environment, and that the hysteresis peak is achieved when the two time scales become comparable. A bell-shaped scaling law of the rate dependence is derived, agreeing quantitatively well with the results of experiments.
\end{abstract}

\section{1. Introduction}

Shape memory alloys (SMAs) have superior shape memory and superelastic properties due to the first order thermo-elastic austenite-to-martensite phase transition (Otsuka and Wayman, 1998). Moreover, many SMAs (e.g., NiTi) are wear resistant and high-damping materials (Qian et al., 2005, 2006; Ortin and Delaey, 2007; Casciati et al., 2008; Heller et al., 2009). In most applications of SMA as a damper, slim SMA components such as long wires, strips and bars are frequently adopted in loading conditions of tension, compression, torsion and even bending. In civil engineering, the cross-sections of SMA bars can be large (with diameter up to 50 mm) to sustain large forces (Dolce et al., 2000). The large damping capacity of a superelastic NiTi SMA bar under a tensile loading-unloading cycle is mainly caused by the large isothermal stress hysteresis between the forward and reverse phase transition plateaus, i.e., a large value of $H_t = \sigma_{AM} - \sigma_{MA}$ as shown in Fig. 1(a) (isothermal case where the loading rate is small and/or heat transfer is strong to keep the specimen’s temperature equal to ambient temperature). The strain range of the full stress plateau is usually defined as the transformation strain $\varepsilon_t$ (here around 4%). So the iso-thermal damping capacity (energy dissipation per unit material volume, i.e., hysteresis loop area) under tension is simply the isothermal stress hysteresis $H_t$ multiplied by the transformation strain. Since the transformation stresses are sensitive to the specimen’s temperature which can easily be changed by the latent heat release/absorption in the forward/reverse phase transformation (see Fig. 1(b)), many experiments have shown that the stress-strain response of the specimen exhibits a strong rate dependence (see Leo et al., 1993; Shaw and Kyriakides, 1995; Bruno et al., 1995; Brinson et al., 2004; Grabe and Bruhns, 2008; He and Sun, 2010a; Zhang et al., 2010; He et al., 2010). It is well recognized that the measured stress strain response and the associated hysteresis for an SMA bar in general cannot be simply taken as pure material properties, as they depend on the loading rate and ambient condition. Such rate dependence of hysteresis (damping capacity) could be a main concern in many applications of the material and therefore worth investigation.

In literature, the reported trend of rate-dependence are not always consistent: some showed an increase in hysteresis with increasing rate (Leo et al., 1993; Tobushi et al., 1998; Pieczyska et al., 2005) while others showed a decrease in hysteresis with increasing rate (Gandhi and Wolons, 1999; Dolce and Cardone, 2001; Heller et al., 2009). In more recent experiments (Piedade et al., 1998; Vitiello et al., 2005; Zhu and Zhang, 2007; Dayananda and Subba Rao, 2008; Zhang et al., 2010; He et al., 2010), it is shown that the rate dependence of hysteresis is actually...
non-monotonic with a hysteresis peak (maximum damping capacity) at an intermediate strain rate which depends on ambient condition. A typical measured non-monotonic rate dependence of a hysteresis-loop in a tensile loading–unloading cycle is shown in Fig. 2 (He et al., 2010). At a very low loading rate (near isothermal condition), the response is characterized by two well-defined forward and reverse transformation stress plateaus (see the upper plateau $\sigma_{AM0} \approx 350$ and the lower plateau $\sigma_{MA0} \approx 180$ MPa at strain rate $\dot{\varepsilon} = 4 \times 10^{-2}/s$ in Fig. 2(a)). When the strain rate is increased (e.g., $\dot{\varepsilon} = 3.0 \times 10^{-7}/s$ in Fig. 2(a)), the forward (reverse) transformation stress is higher (lower) than $\sigma_{AM0}$ ($\sigma_{MA0}$) due to the self-heating (self-cooling) of the specimen by the latent heat release (absorption); so the hysteresis increases with increasing $\dot{\varepsilon}$. However, if $\dot{\varepsilon}$ is further increased (e.g., $\dot{\varepsilon} = 3 \times 10^{-3}/s$ in Fig. 2(a)), the released heat in the loading process will not have enough time to transfer out completely and part of the heat will be carried to the unloading process, making the specimen’s temperature at the start of the reverse phase transition much higher than the ambient temperature (see the unloading curve at the strain rate $\dot{\varepsilon} = 3 \times 10^{-2}/s$). This is evidenced by the much higher value of the reverse transformation stress than the isothermal one (lower plateau $\sigma_{MA0}$). Such high reverse transformation stresses reduce the hysteresis loop area significantly. Therefore, in the whole strain rate range (the tests of 23 different strain rates in the range of $10^{-5}$–$10^{-3}/s$ in Fig. 2(b)), the area of the hysteresis loop (damping capacity $D$) varies non-monotonically with the strain rate, reaching its maximum (peak) at a critical strain rate ($\dot{\varepsilon}_{critical} \approx 3 \times 10^{-3}/s$). The critical strain rate for the hysteresis peak depends on the ambient condition: it increases with the increase in the flow velocity of ambient air (He et al., 2010). This non-monotonic rate dependence of hysteresis was observed not only in NiTi polycrystals, but also in single crystal SMAs, e.g., CuZnAl and CuAlNi (Van Humbeeck and Delaey, 1981; Yin, 2011).

The main physical reason for the above rate dependence of the hysteresis is that the phase transition or the deformation of the material deviates away from isothermal condition due to the release/absorption of the latent heat and the heat transfer with the environment (see Fig. 1(b)). When the applied strain rate is high, the high rate of latent heat release/absorption causes a swift temperature variation of the material, which leads to the significant variation in the applied stress and the stress–strain response since the transformation stress of the material is very sensitive to temperature. Many researches (Shaw and Kyriakides, 1995; Grabe and Bruhns, 2008, among many others) have shown that the above

**Fig. 1.** Schematic stress–strain response and damping capacity of SMA in isothermal tension (very low strain rate and/or strong heat transfer) (a); and in non-isothermal tension (high strain rate and/or weak heat transfer) (b).

**Fig. 2.** Experimental observations (He et al., 2010) of the rate dependence of the stress–strain curves (a) and the hysteresis-loop area (b) in the first loading-unloading cycle.
observed rate effect is in fact a temperature effect. According to the heat transfer theory, the temperature variation is governed by both the rate of latent-heat release/absorption and the rate of heat exchange (here mainly convection) between the specimen and the environment. It is the competition between the loading (heat release) and the heat transfer that leads to the experimentally observed non-monotonic rate dependence of the hysteresis. Due to the strong nonlinearity of the SMA’s constitutive behavior, complicated deformation mode in tension and the effect of the thermo-mechanical coupling, only limited studies (Vitiello et al., 2005; Zhu and Zhang, 2007; Morin et al., in press) have been performed to quantify the rate-dependent hysteresis so far. Theoretical modeling and analytic relationship between the hysteresis and the governing factors (such as external applied strain/loading rate, ambient condition and material thermal and mechanical properties) have not been available in literature. In order to achieve a simple and straight forward understanding, the key issue is how to handle the deformation heterogeneity and the resulting evolving temperature field even in the 1D structure like a bar.

From a fundamental point of view, both deformation and temperature fields in single and polycrystalline materials during martensitic phase transitions are intrinsically heterogeneous due to the existence of domains and interfaces (Shaw and Kyriakides, 1995; Zhang et al., 2000; Zhang et al., 2010; Sun et al., 1994; Sun and Li, 2002). There are three aspects of the transformation process to which we have to pay attention. First, it has been well known that for NiTi polycrystalline SMA under tension, both deformation and temperature fields are macroscopically inhomogeneous. The length scale of such deformation heterogeneity – the domain spacing – is in fact due to the multiple domain nucleation events in which the non-uniformity of the temperature field play critical role. Both simulation and modeling (Iadicola and Shaw, 2004; He and Sun, 2010a) have demonstrated that the domain spacing is governed by the strain rate (heat release rate) and the conduction of the non-uniform temperature field within the specimen. Second, the overall stress–strain response and the associated damping capacity are mainly governed by the release and absorption of the latent heat and heat transfer to the environment. Finally, the transfer of heat from a slim circular bar (of radius \( R \) and length \( L \) with \( L > R \)) to the outer environment consists of heat to ambient air convection to ambient air (through the circumference surface) and heat conduction to the grips of test machine (through the two ends). We can compare the characteristic time scales of heat convection \( L^2/k \) and conduction \( L/c_2 \), here \( k \) is the heat conductivity and \( c_2 \) is the heat capacity per unit volume. For a typical NiTi bar (in stagnant air) with \( k = 18.3 \text{ J/(m s K)} \), \( L = 175 \text{ mm} \), \( T = 100 \text{ mm} \), \( h = 6.5 \text{ J/(m² s K)} \), the ratio of the two time scales is less than 0.25. This means that the heat convection is dominant in heat exchange between the specimen and environment. For the internal heat conduction between the martensite/austenite bands in the specimen, the characteristic heat conduction time (which depends on domain spacing) is much shorter than the phase transition (loading) time so that the local temperature heterogeneity may be a minor factor in determining the global stress–strain responses. Here, we directly employ the classical lumped system analysis method (Cotta and Mikhailov, 1997) and use the lumped temperature (i.e., average temperature of the bar) to characterize the heat convection process and to determine the overall stress–strain responses. As shown in the following of the paper, it indeed helped us to turn a complicated partial differential equation into a simple “one body equation” and to get extremely simple results.

The aim of this paper is, using the simplest model possible, to obtain an analytical expression of the rate dependent stress hysteresis, so that quantitative predictions of and a unified view on the rate dependence and the ambient effect (effect of heat exchange) can be obtained. To facilitate direct comparisons with experiments and without losing generality, we consider a slim SMA bar under a tensile loading–unloading cycle. In Section 2, we give the basic assumptions and simplifications in the modeling. By solving the heat transfer equations with heat sources/sinks, the temperature history of the bar in the cycle can be obtained. Based on the temperature dependence of the transformation stresses, an analytical expression of the rate-dependent stress hysteresis is derived in Section 2. The predictions of the model are discussed and compared with the results of experiments in Section 3. The implication of the model prediction and the conclusions are given in Section 4.

2. Theoretical model

We consider an SMA bar (radius \( R \)) undergoing a stretching cycle (“OABCDO” in Fig. 1(b)) at a strain rate \( \dot{\varepsilon} \). In the determination of the hysteresis loop area, the coupling between the phase transitions and the average temperature variation of the bar in the cycle mainly involves the following three stages: (1) forward, exothermic phase transition stage (stage “AB” with the characteristic transformation strain \( \varepsilon_f = \varepsilon_f - \varepsilon_r \)) in which latent heat is released to raise the bar’s temperature and at the same time some of the heat is transferred to the convective environment; (2) elastic unloading stage (“BC” with elastic unloading strain \( \varepsilon_u = \varepsilon_f - \varepsilon_r \)) in which some heat is transferred to the environment and the bar’s temperature is reduced; (3) reverse, endothermic phase transition stage (“CD” with the same transformation strain \( \varepsilon_f \)) in which latent heat is absorbed by the material, and at the same time some heat is exchanged with the convective environment. To derive a simple analytical solution, the following assumptions are made in the model:

1. \( \dot{\varepsilon} \) and \( \dot{\varepsilon}_0 \) are constants, independent of temperature and strain rate;
2. Heat release/absorption during elastic deformations (e.g., stages “OA” and “BC”) is ignored, i.e., no heat source/sink during elastic deformation;
3. Ignore the local temperature heterogeneity and use the average temperature of the bar (i.e., lumped temperature) to characterize the heat convection process and to determine the overall stress;
4. Only consider the heat exchange between the slim bar (length \( L > \) radius \( R \)) and the environment by the heat convection to the ambient air (through the circumference surface).

With the above assumptions, the evolution of the bar’s average temperature \( T(t) \) and the associated stress hysteresis in the stretching cycle can be determined as follows.

2.1. Non-isothermal stress hysteresis and its loading rate dependence

According to the Clausius–Clapeyron relationship, the forward (Austenite → Martensite) and reverse (Martensite → Austenite) transformation stresses, \( \sigma_{AM} \) and \( \sigma_{MA} \), of the SMA bar can be well approximated as linear functions of the bar’s temperature \( T \) as (Shaw and Kyriakides, 1995; Vitiello et al., 2005; Zhu and Zhang, 2007)

\[
\begin{align*}
\sigma_{AM}(T) &= \sigma_{AM}(T_0) + \frac{b_{AM}}{T_0} \cdot (T - T_0), \\
\sigma_{MA}(T) &= \sigma_{MA}(T_0) + \frac{b_{MA}}{T_0} \cdot (T - T_0),
\end{align*}
\]

where \( b_{AM} \) and \( b_{MA} \) are material constants; \( T_0 \) is a reference temperature, taken as the ambient temperature here. As an important measure of the material’s dissipative property, the stress hysteresis \( H \) is defined as the difference between \( \sigma_{AM} \) (the average of \( \sigma_{AM} \) over
$\varepsilon_T$ and $\sigma_{\Delta m}$ (the average of $\sigma_{MA}$ over $\varepsilon_T$) in forward (AB) and reverse (DC) phase transitions (see Fig. 1(b))

$$H = \frac{1}{\varepsilon_T} \left( \int_{AB} [\sigma_{MA}(\theta)] d\varepsilon - \int_{DC} [\sigma_{MA}(\theta)] d\varepsilon \right) = H_0$$

$$+ \frac{1}{\varepsilon_T} \left\{ \left[ b_{MA} \cdot \int_{0}^{\varepsilon_T} ((\theta - \theta_0) d\varepsilon) \right]_{AB} - \left[ b_{MA} \cdot \int_{0}^{\varepsilon_T} ((\theta - \theta_0) d\varepsilon) \right]_{CD} \right\}$$

$$= H_0 + \Delta H$$

or

$$\Delta H = H - H_0 = \frac{1}{\varepsilon_T} \left( \left[ b_{MA} \cdot \int_{0}^{\varepsilon_T} ((\theta - \theta_0) d\varepsilon) \right]_{AB} - \left[ b_{MA} \cdot \int_{0}^{\varepsilon_T} ((\theta - \theta_0) d\varepsilon) \right]_{CD} \right),$$

where

$$H_0 = \sigma_{\Delta m}(\theta_0) - \sigma_{\Delta m}(\theta_0)$$

and $\varepsilon_T = \frac{\varepsilon_T}{\varepsilon_T}$

$t_1$ is the time for forward (or reverse) phase transition. It is seen that $\Delta H$ can be split into two parts: $H_0$ and $\Delta H$. The first part $H_0$ is the isothermal stress hysteresis at the reference temperature $\theta_0$ (ambient temperature). The second part $\Delta H$ is the non-isothermal contribution to the hysteresis. When the strain rate $\dot{\varepsilon}$ is very small (i.e., $\dot{\varepsilon} \to 0$, $\varepsilon_T \to \infty$), there is sufficient time for the latent heat to transfer to the environment so that isothermal condition prevails $(\theta = \theta_0)$, we have $\Delta H \to 0$ and $H \to H_0$. It is seen that $\Delta H$ depends on the material’s temperature history which is controlled by the external loading rate $\dot{\varepsilon}$ (or $t_1$), the material thermal properties and the convective environment as shown below.

2.2. Temperature history in the first loading–unloading cycle

2.2.1. Temperature variations in forward phase transition (stage “AB”)

The heat conduction equation for the bar now can be simplified as one-body equation with constant heat release rate (governed by loading rate during the forward phase transition (e.g., see Bruno et al., 1995))

$$\dot{\theta} = \frac{d\theta}{dt} = q_{MA} - \frac{2h}{R} (\theta - \theta_0),$$

where $\theta$ represents the average temperature of the bar; $R$, $h$, and $q_{MA}$ are, respectively, the bar radius, the convection coefficient, the heat capacity (per unit material volume) and the heat release (generation) rate. The total released heat $l_{MA}$ (per unit material volume) in this process (Austenite $\rightarrow$ Martensite transition) is the sum of the latent heat $l_0$ and the mechanical dissipation heat $H_0 - \varepsilon_T / 2$ (here it is assumed that half of the mechanical dissipation (hysteresis loop in Fig. 1(a)) occurs in the loading process while the other half occurs in unloading process), i.e., $l_{MA} = l_0 - \varepsilon_T / 2 - l_0$. So the heat generation rate $q_{MA} = \frac{h}{R} \dot{\theta}$, where $\dot{\theta}$ is the loading rate for the transformation. By normalization, Eq. (3a) becomes

$$\frac{d\Delta \theta}{dt} = \frac{\Delta \theta}{t_1} - \frac{t_1}{\varepsilon_T} \Delta \theta,$$

where $\Delta \theta = \frac{\dot{\theta}}{\dot{\varepsilon}} - \dot{\theta}_0$ with $\dot{\theta}_0 = \frac{\dot{\theta}_0}{\varepsilon_T}$, $t = \int_{0}^{\varepsilon_T} t_1$ with $t_0 = \frac{\varepsilon_T}{\varepsilon_T}$ and $t_1 = \frac{\varepsilon_T}{\varepsilon_T}$.

It is noted that $t_0$ is the characteristic relaxation time of the heat convection between the specimen and the environment (experimental measurements of the characteristic heat transfer timescale in different ambient conditions can be found in He et al. (2010)). Using the initial condition $\dot{\theta}_{t=0} = \theta_0$ (i.e., $\Delta \theta_{t=0} = 0$), the solution of Eq. (3b) provides the temperature evolution of the bar in the stage “AB” ($t \in [0, t_1]$ or $t \in [0, \frac{\varepsilon_T}{\varepsilon_T}]$):

$$\Delta \theta = \frac{t_0}{t_1} \cdot \left( 1 - e^{-\frac{t_0}{t_1}} \right)$$

or expressed as

$$\theta = \theta_0 + \Delta \theta = \theta_0 - \frac{t_0}{t_1} \cdot \theta_{MA} \cdot \left( 1 - e^{-\frac{t_0}{t_1}} \right)$$

From Eq. (4b), the bar’s temperature at the end of the stage “AB” (i.e., $t = t_1$, point B) is

$$\theta_B = \theta_0 + \frac{t_0}{t_1} \cdot \theta_{MA} \cdot \left( 1 - e^{-\frac{t_0}{t_1}} \right)$$

or

$$\Delta \theta_B = \theta_B - \theta_0 = \frac{t_0}{t_1} \cdot \theta_{MA} \cdot \left( 1 - e^{-\frac{t_0}{t_1}} \right).$$

Using the material properties of NiTi (Table 1) (He et al., 2010), the rate dependence of $\Delta \theta_B$ (Eq. (4c)) is plotted in Fig. 3 in terms of $t_0 / t_1$. It is seen that $\Delta \theta_B$ increases monotonically with decreasing $t_0 / t_1$ (i.e., increasing $\dot{\varepsilon}$).

2.2.2. Temperature variations in elastic unloading of martensite (stage “BC”)

In this stage ($t \in [0, t_1]$), the time used to elastically unload the martensite by the amount of strain $\varepsilon_T$ (see Fig. 1(b)) is $t_0 = \frac{\varepsilon_T}{\varepsilon_T}$. Since there is no heat release (no phase transition), the temperature variation of the bar with time is only caused by heat convection:

$$\lambda \cdot \frac{d\theta}{dt} = -\frac{2h}{R} \theta$$

With the initial condition $\dot{\theta}_{t=0} = \theta_B$, the solution of Eq. (5) is

$$\theta = \theta_0 + (\theta_0 - \theta_B) \cdot e^{-\frac{t_0}{t_1}}.$$ 

Thus, at the end of the elastic unloading process, point C, the bar’s temperature, is

$$\theta_C = \theta_0 + (\theta_0 - \theta_B) \cdot e^{-\frac{t_0}{t_1}} \text{ or } \Delta \theta_C = \Delta \theta_B \cdot e^{-\frac{t_0}{t_1}}$$

where $t_0 = \frac{\varepsilon_T}{\varepsilon_T}$ for NiTi polycrystal under a tensile loading (Table 1) has been used for the purpose of simplicity. Eq. (6b) is plotted in Fig. 3. It is seen that $\Delta \theta_C$ also increases monotonically with decreasing $t_0 / t_1$ (i.e., increasing $\dot{\varepsilon}$), but it is always lower than $\Delta \theta_B$ due to heat convection.

2.2.3. Temperature variations in reverse phase transition (stage “CD”)

The heat generated per unit material volume, $l_{MA}$, in this stage of Martensite $\rightarrow$ Austenite phase transition is the sum of the negative latent heat $-l_0$ (i.e., to absorb heat) and the positive heat from mechanical dissipation, i.e., $l_{MA} = H_0 - \varepsilon_T / 2 - l_0$. The governing equation for the temperature variation in the reverse phase transition ($t \in [0, t_1]$) is

$$\dot{\theta} = \frac{d\theta}{dt} = q_{MA} - \frac{2h}{R} (\theta - \theta_0)$$

where $q_{MA} = \frac{h}{R} \dot{\theta}$. With the initial condition $\dot{\theta}_{t=0} = \theta_C$, the solution of Eq. (7) is

$$\theta = \theta_0 - \frac{t_0}{t_1} \cdot \theta_{MA} + \left( \theta_C - \theta_0 + \frac{t_0}{t_1} \cdot \theta_{MA} \right) \cdot e^{-\frac{t_0}{t_1}},$$

where $\theta_{MA}$ is the characteristic temperature scale of the heat source in the reverse phase transition (note that $\theta_{MA} > 0$ as $l_0 > H_0 - \varepsilon_T / 2$ for the NiTi SMA considered, see Table 1). The bar’s temperature at the end of the reverse phase transition (i.e., point D) is

$$\theta_D = \theta_0 - \frac{t_0}{t_1} \cdot \theta_{MA} + \left( \theta_C - \theta_0 + \frac{t_0}{t_1} \cdot \theta_{MA} \right) \cdot e^{-\frac{t_0}{t_1}} \text{ or }$$

$$\Delta \theta_D = \frac{t_0}{t_1} \cdot \theta_{MA} + \left( \Delta \theta_C + \frac{t_0}{t_1} \cdot \theta_{MA} \right) \cdot e^{-\frac{t_0}{t_1}}.$$

$\Delta \theta_D$ is plotted in Fig. 3 for comparison with $\Delta \theta_B$ and $\Delta \theta_C$. It is important to note that the theoretically predicted $\Delta \theta_D$ depends
3. Discussion

3.1. Rate dependence of hysteresis of single crystal SMA with negligible static hysteresis ($H_0 \approx 0$)

Some superelastic single crystal SMA (e.g., CuZnAl and CuAlNi) has negligible isothermal hysteresis as shown in Fig. 4(a) (i.e., $H_0 \approx 0$, also see Van Humbeeck and Delaey, 1981; Sun et al., 1999; Zhang et al., 2000). In this case, the isothermal forward and reverse phase transformation stresses can be approximated as being equal (i.e., $\sigma_{AM}(H_0) = \sigma_{MA}(H_0)$) and the constants $b_{AM}$, $b_{MA}$, $\epsilon_{AM}$, $\theta_{AM}$ and $\theta_{MA}$ become:

$$\Delta H = H - H_0 = \frac{f_0}{f_1} \left( b_{AM} \cdot \theta_{AM} + b_{MA} \cdot \theta_{MA} \right) \cdot \frac{1}{2} \left[ 1 - \frac{f_0}{f_1} \left( 1 - e^{-\frac{t}{t_T}} \right) \right]$$

$$- \frac{f_1}{f_0} b_{AM} \cdot \theta_{AM} \cdot e^{\frac{t}{t_T}} \left( 1 - e^{-\frac{t}{t_T}} \right)^2,$$

(9)

where $\Delta$ is the change in stress with the environment. Furthermore, it is easy to show that the $\Delta H$ relationship is a bell-shaped curve with a peak around $t_f/t_0 = 1$. In the following, we discuss the implication of Eq. (9) for some SMA single crystals with a very small isothermal hysteresis (i.e., $H_0 \approx 0$) and for NiTi polycrystalline SMAs with a large isothermal hysteresis (large $H_0$).

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$$\Delta H = H - H_0 = \frac{f_0}{f_1} \left( b_{AM} \cdot \theta_{AM} + b_{MA} \cdot \theta_{MA} \right) \cdot \frac{1}{2} \left[ 1 - \frac{f_0}{f_1} \left( 1 - e^{-\frac{t}{t_T}} \right) \right]$$

$$- \frac{f_1}{f_0} b_{AM} \cdot \theta_{AM} \cdot e^{\frac{t}{t_T}} \left( 1 - e^{-\frac{t}{t_T}} \right)^2,$$

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$$\Delta H = H - H_0 = \frac{f_0}{f_1} \left( b_{AM} \cdot \theta_{AM} + b_{MA} \cdot \theta_{MA} \right) \cdot \frac{1}{2} \left[ 1 - \frac{f_0}{f_1} \left( 1 - e^{-\frac{t}{t_T}} \right) \right]$$

$$- \frac{f_1}{f_0} b_{AM} \cdot \theta_{AM} \cdot e^{\frac{t}{t_T}} \left( 1 - e^{-\frac{t}{t_T}} \right)^2,$$

(9)

where $\Delta$ is the change in stress with the environment. Furthermore, it is easy to show that the $\Delta H$ relationship is a bell-shaped curve with a peak around $t_f/t_0 = 1$. In the following, we discuss the implication of Eq. (9) for some SMA single crystals with a very small isothermal hysteresis (i.e., $H_0 \approx 0$) and for NiTi polycrystalline SMAs with a large isothermal hysteresis (large $H_0$).
As shown in Fig. 4(b), Eq. (11) has a bell shape, showing a non-monotonic dependence of $\Delta H$ on the time ratio $t_f/t_0$. The two extremes of the curve at $\frac{t_f}{t_0} \to 0$ and $\frac{t_f}{t_0} \to \infty$ represent the isothermal and the adiabatic limits of $\Delta H$, respectively, where the non-isothermal contribution $\Delta H \to 0$. In the isothermal limit ($\frac{t_f}{t_0} \to 0$), there is sufficient time for the heat exchange between the bar and the environment, so the bar's temperature is always equal to the ambient temperature and $\Delta H = 0$. In the adiabatic limit ($\frac{t_f}{t_0} \to \infty$), no heat exchange between the bar and the environment is allowed, so the latent heat makes the bar's temperature higher than the ambient temperature and both $\sigma_{AM}$ and $\sigma_{MA}$ increase by the same amount, i.e., the net thermal effect on the hysteresis is zero. At a certain intermediate strain rate ($\frac{t_f}{t_0} \approx 1$), the hysteretic $H$ for a given material (fixed $b$, $l_0$, and $\lambda$) reaches a peak. Such predicted non-monotonic rate dependence of the hysteresis in single crystals has been observed in experiments on CuZnAl crystals (van Humbeek and Delaey, 1981) and recently on CuAlNi (Yin, 2011). Finally, it should be noticed that Eq. (11) is a good approximation as far as the material has a relatively small isothermal hysteresis, i.e., $\frac{H_{el}}{H_{i}} \ll 1$. For the material with a large isothermal hysteresis (such as most polycrystalline SMAs), the effect of the mechanical dissipative work ($H_{el}$) on the temperature variation must be considered as in Eq. (9).

3.2. Rate dependence of hysteresis of NiTi polycrystalline SMA with large static hysteresis

To compare the predicted hysteresis with experiments, we roughly estimate the area of the hysteresis loop (damping capacity $D$) by multiplying the stress hysteresis $H$ with the transformation strain $\varepsilon_T$ as:

$$D = H \cdot \varepsilon_T.$$  

(12)

In experiments, it is more convenient to use the area of the hysteresis loop ($D$) normalized by transformation strain to characterize the average stress hysteresis of a stretched SMA bar. Substituting Eq. (9) into Eq. (12) and using the material properties of a typical polycrystalline NiTi (Table 1), the relation between the damping capacity $D$ and $\frac{t_f}{t_0}$ can be obtained and is plotted in Fig. 5 (solid line). It is seen that the hysteresis varies non-monotonically with $\frac{t_f}{t_0}$ and reaches a peak when the two time scales are comparable ($\frac{t_f}{t_0} \approx 1$). This prediction is valid for different convective environments. For the purpose of comparison, the experimental data obtained in three different convective environments (still air and flowing air of velocities $v = 2$ and $17$ m/s (He et al., 2010) which are characterized by different characteristic heat transfer times $t_0$, see Table 1) are also plotted in Fig. 5. It is seen that, without any fitting parameters, the present theoretical predictions well captured both the critical strain rate and the value of damping peak from experiments. The differences between the experiments and the predictions could be due to the assumptions and simplifications made in the modeling, which are to be refined in the future.

3.3. Critical condition for hysteresis peak

In many engineering practices, such as device design for seismic damping, vibration control and MEMS, it is important to determine the values of the external control parameters such as strain rate in order to achieve maximum damping. Using the present model, the critical loading time ($\frac{t_f}{t_0}_{critical}$ or $\tilde{t}_{critical}$) to achieve the hysteresis peak can be estimated from the following:

$$I \equiv \frac{(t_f \frac{t_f}{t_0}_{critical})}{t_0} = \frac{1}{\lambda} = \frac{2 h \cdot \varepsilon_T}{\lambda \cdot R \cdot \tilde{t}_{critical}},$$  

(13a)

from which we have

$$\tilde{t}_{critical} = \frac{2 h \cdot \varepsilon_T}{\lambda \cdot R}.$$  

(13b)

It is seen that the critical strain rate $\tilde{t}_{critical}$ for the hysteresis peak can be directly calculated from material properties (transformation strain $\varepsilon_T$ and heat capacity $\lambda$), specimen's geometry (bar radius $R$) and the heat convection coefficient $h$ of the environment.

3.4. Stress hysteresis in isothermal and adiabatic conditions

For isothermal condition ($t_f/t_0 \to \infty$), Eq. (9) reduces to

$$\Delta H_{i} = 0.$$  

(14a)

In that case (the loading rate is very low and/or the heat convection is very strong) the specimen's temperature is always equal to the ambient temperature and the stress hysteresis is only the difference between the upper and lower isothermal stress plateaus (Fig. 1(a)). On the other hand, by using L'Hospital's rule, Eq. (9) for the adiabatic condition ($t_f/t_0 \to 0$) can be obtained as:

$$\Delta H_{a} = \frac{1}{2} \left[ (b_{AM} - b_{MA}) \cdot \theta_{AM} + b_{MA} \cdot (\theta_{MA} - \theta_{AM}) \right]$$

$$= \frac{1}{2} \left[ (b_{AM} - b_{MA}) \cdot \frac{H_{el}}{\lambda} - b_{MA} \cdot H_{el} \cdot \varepsilon_T \right]$$  

(14b)

For most NiTi polycrystals, $b_{AM} > b_{MA} > 0$ (Table 1). Therefore $\Delta H_{a_{i\frac{t_f}{t_0} \to 0}} < 0$, which means that the adiabatic hysteresis is less than the isothermal hysteresis (i.e., $H_{a_{i\frac{t_f}{t_0} \to 0}} < H_{i_{\frac{t_f}{t_0} \to 0}}$) which has been experimentally verified in a recent paper by Zhang et al. (2010).

4. Summary and conclusions

Different from SMA's intrinsic static isothermal stress–strain constitutive relations which are rate-independent, the hysteretic response of superelastic SMA structures or structure components
such as a rod or tube under most external loadings strongly depends on the loading rate, specimen's geometry (such as the radius of rod) and ambient conditions. Physically, such a strong rate dependence of the hysteresis is caused by the non-isothermality of the phase transition due to the release/absorption of heat and heat transfer with the environment. For long thin SMA structure members like rod, strip and tube under tension, it can be shown that the convection is the dominant mode of heat transfer with the environment and that we can ignore the heterogeneity of the temperature field in these members and use the average temperature (i.e., lumped temperature) to characterize the heat convection process and to determine the overall stress. By solving the lumped convective heat transfer equation and employing the temperature dependence of the SMA's transformation stresses, we have studied the rate dependent stress hysteresis phenomenon using the simplest model possible in one tensile loading-unloading cycle. The analytical expression of the rate dependence is obtained. We found that such a rate dependence involves the coupling among the temperature-dependence of transformation stress, the effects of latent-heat, mechanical dissipative heat and heat exchange with the environment. This rate dependence is intrinsically a thermal phenomenon, governed by the specimen's geometry and the competition of phase-transition time \( t_f \) (or latent-heat release/absorption time as controlled by the external applied loading rate) and the time \( t_h \) of heat exchange with the environment which is determined by the specimen's geometry and the ambient condition. The results obtained here for the first loading-unloading cycle can be extended to the hysteresis loop (damping capacity) in steady-state cyclic loading conditions (He and Sun, 2010b). We would like to point out that the concept and results obtained from the present 1D structure under tension may be applied to more general cases of 2D and 3D SMA structures under other loading conditions (e.g., compression and torsion) as well. That is, each structure in a given ambient has a characteristic heat transfer time; when the external driving time is close to this heat transfer time, the hysteresis (damping capacity) will be maximized.

The key theoretical results of the paper are listed as follows:

- In any given convective ambient condition characterized by \( t_h \), the stress hysteresis of an SMA bar varies non-monotonically with \( \kappa (= \frac{t_f}{t_h}) \), following a bell-shaped scaling law.
- The stress hysteresis peak is achieved at a critical strain rate \( \dot{\varepsilon}_c \), for which the phase transition time \( (t_f) = \dot{\varepsilon}_c \) is close to the characteristic heat-transfer time \( t_h \) of the bar-ambient system.
- The critical strain rate \( \dot{\varepsilon}_c \) can be directly calculated from material properties (transformation strain \( \dot{\varepsilon}_t \) and heat capacity \( \lambda \)), specimen's geometry (bar radius \( R \)) and the heat convection coefficient \( h \) of the ambient.

The above theoretical results are discussed for the cases of single crystal SMA with a very small isothermal hysteresis and polycrystalline SMA with a large isothermal hysteresis. The theoretical predictions are compared with the measured stress hysteresis data of polycrystalline NiTi SMA over the strain-rate range of \( 10^{-3} \text{ to } 10^{-1} \text{s}^{-1} \) and in three ambient conditions: still air and flowing air with velocities of 2 m/s and 17 m/s. The theoretical predictions well captured the experimental results without any fitting parameters.

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