# CHAPTER FIVE

# TORSION, UNSYMMETRIC BENDING AND SHEAR CENTER

5.1 Torsion of noncircular members and thin-walled hollow shafts (MECH101, pp.163-169)

5.2 Unsymmetric bending of beams and the principal centroidal axes of the cross section (MECH 101, pp.242-251)

5.3 Unsymmetric loading of thin-walled members, Shear center (MECH 101,pp.320-327)

**Review and Summary**

## 5.1 Torsion of noncircular members and thin-walled hollow shafts

- Torsion of noncircular members



(a)        (b)        (c)
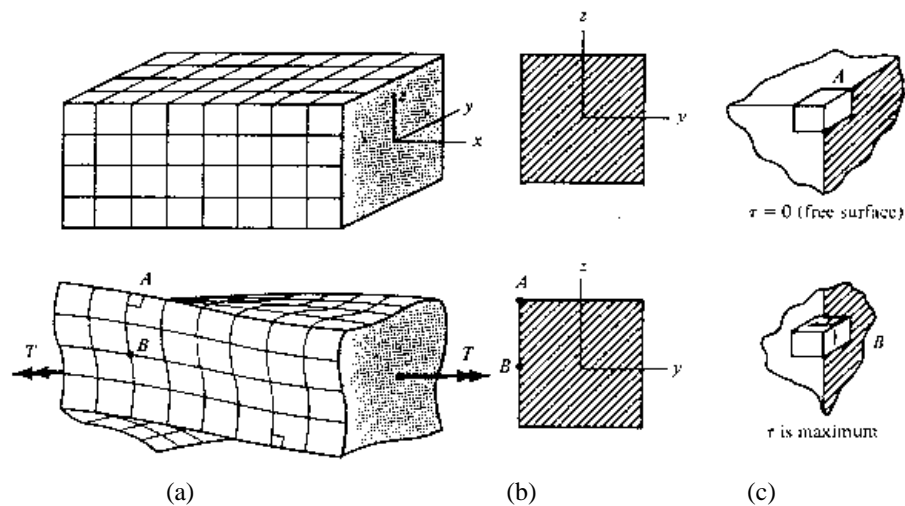
**FIGURE8.12** (a) Torsion of a bar of square cross section, showing distortion of a rectangular grid on the lateral surfaces.(b)Cross section remain square.(Thus distortion of the lateral surface grid is caused by x-direction warping motion.) (c) State of stress at points A and B under torque T.
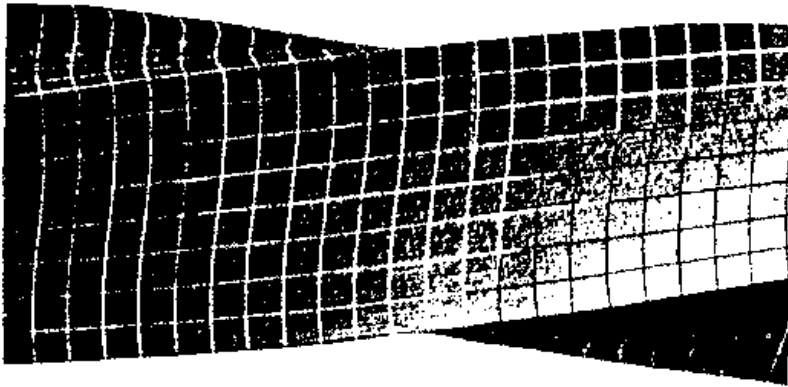
Fig. 3.44



On the corner of the cross section :

$\tau_{yx}=0 \qquad \tau_{yz}=0 \qquad \tau_{zx}=0 \qquad \tau_{zy}=0$

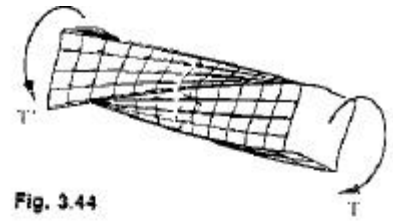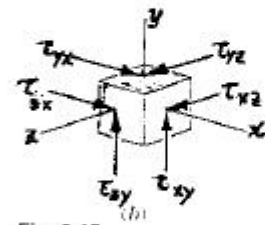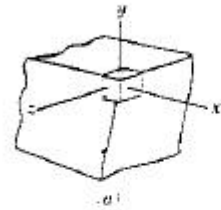$\tau_{xy}=0 \qquad\qquad\qquad \tau_{xz}=0$
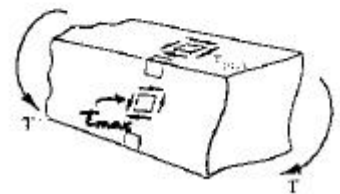


Fig. 3.45
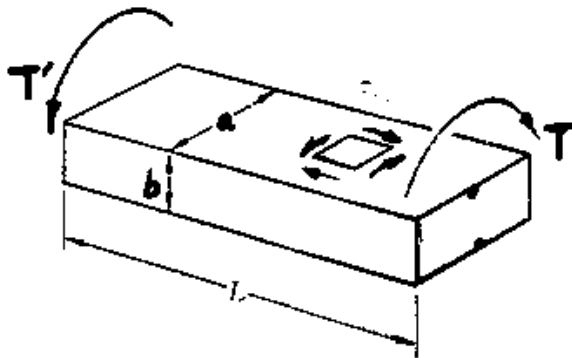


Fig. 3.46

- ## Shearing in uniform rectangular cross section

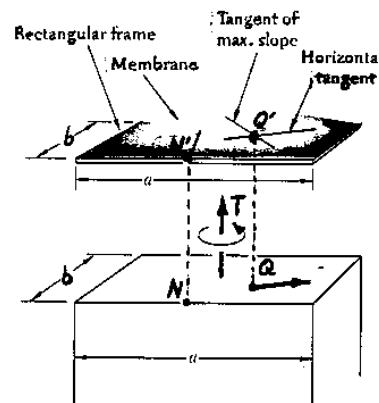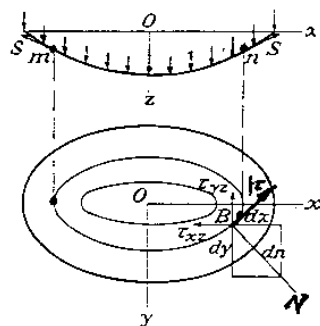

**Table 3.1** Coefficient for Rectangular Bars in Torsion

| A/b | $C_1$ | $C_2$ |
|---|---|---|
| 1.0 | 0.208 | 0.1406 |
| 1.2 | 0.219 | 0.1661 |
| 1.5 | 0.231 | 0.1958 |
| 2.0 | 0.246 | 0.229 |
| 2.5 | 0.258 | 0.249 |
| 3.0 | 0.267 | 0.263 |
| 4.0 | 0.282 | 0.281 |
| 5.0 | 0.291 | 0.291 |
| 10.0 | 0.312 | 0.312 |
| ∞ | 0.333 | 0.333 |

$$t_{max} = \frac{T}{c_1 ab^2}$$

$$f = \frac{TL}{c_2 ab^3 G}$$

For a/b>=5 , $c_1=c_2=1/3 \ (1-0.630b/a)$

- ## Distribution of shearing stress---membrane analogy



* A homogeneous elastic membrane attached to a fixed frame and subjected to a uniform pressure on one side--- analog of the bar in torsion: The shearing stress $\tau$ will have the same direction as the horizontal tangent to the membrane at Q', and its magnitude will be proportional to the maximum slope. The applied torque will be proportional to the volume between the membrane and the plane of the frame.

# ● Shearing stress in thin-walled members
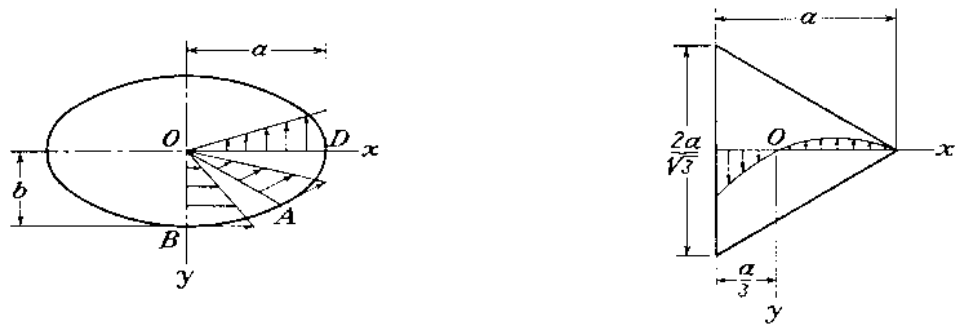
<1> Regular cross section

<2>Thin-walled members
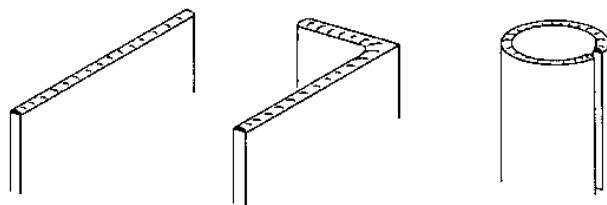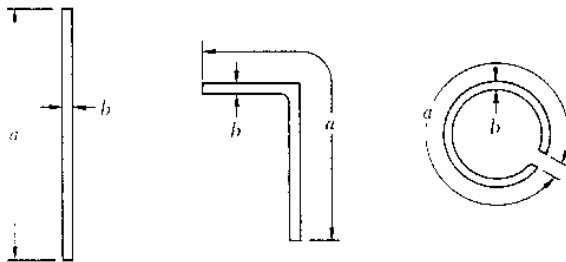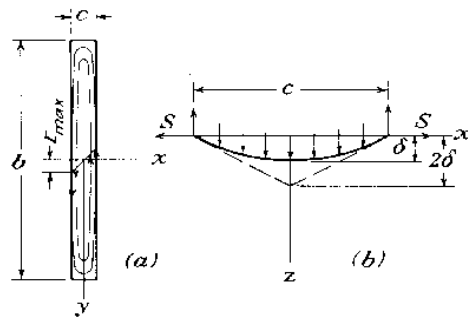
- ## Torsion of thin-walled hollow shafts
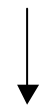
(1)    stress analysis

Consider a hollow cylindrical member of non-circular cross section, equilibrium of the element AB requires



Fig. 3.50

$F_A = F_B$

$$t_A t_A \Delta x = t_b t_b \Delta y$$

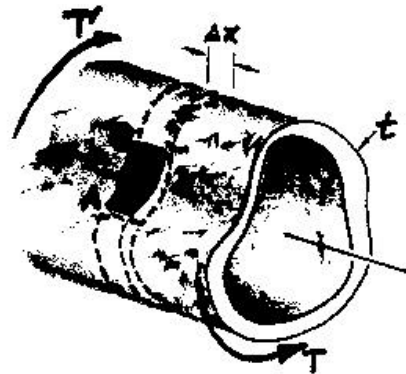(by using shear equivalent)
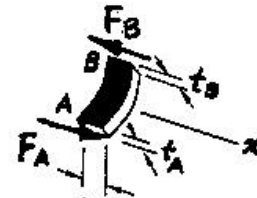
$q = \tau t = $ constant

shear flow



Fig. 3.51

Analogy: (1) the distribution of shear stress $\tau$ in the transverse section of a thin-walled hollow shaft

　　　　(2) the distribution of the velocities v in water flowing through a closed channel of unit depth and variable width.
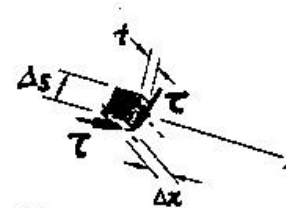


Fig. 3.52

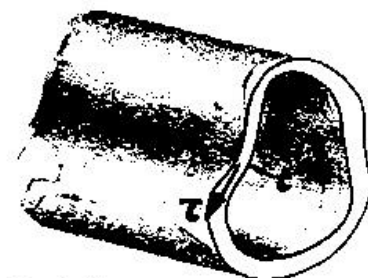Flow in channel ⟺ Shear force in
　q=vt　　　　　the thin-wall
　　　　　　　　　q=τt



Fig. 3.53

v ⟷ τ

(2) Formula for shearing stress and angle of twist

Shearing Stress:

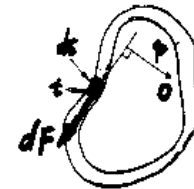- The shear force dF on a small element ds is
  dF=τdA=τ(tds)=qds



Fig. 3.54

- The moment $dM_0$ of dF about an arbitrary point O is
  $dM_0$=pdF=pqds=q(pds)

- The moment $dM_0$ of dF about on arbitrary point O is
  $dM_0$=pdF=pqds=q(pds)
  where pds equals twice the area element da of tringle:
  $dM_0$=q(2da)



Fig. 3.55

- The torque T is
  $$T = \oint dM_0 = \oint q(2da) = 2qA$$
  A----the area bounded by the center line of wall cross section



Fig. 3.56

- The shear stress τ
  $$t = \frac{q}{t} = \frac{T}{2tA}$$
  (circular tube, an special case)

- The angle of twist
  $$f = \frac{TL}{4A^2G} \oint \frac{ds}{t}$$

- Example
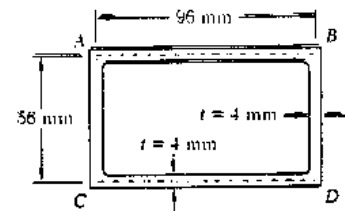


Fig. 3.58

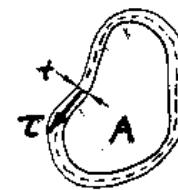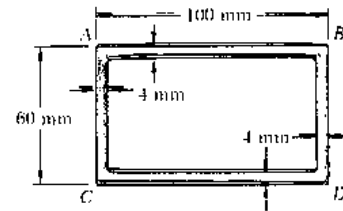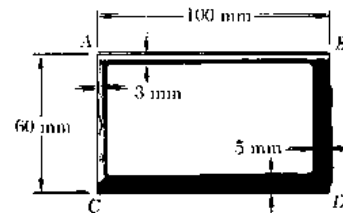

Fig. 3.57

<1> Determine the shear stress in uniform and variable wall thickness

$$t = \frac{T}{2tA} \quad \text{(T is given)}$$

<2> Using $\tau=40$Mpa, determine the largest torque which may be applied to each of the brass bars and tube.



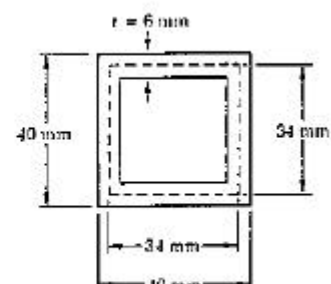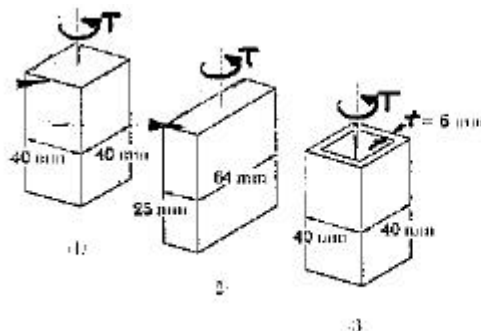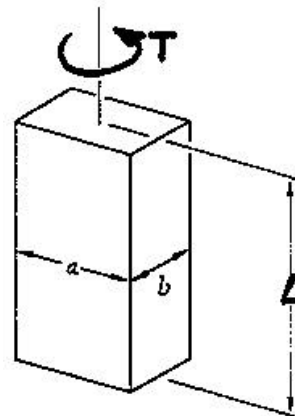a) Bar with square cross section

$\tau_{max}=T/c_1ab^2$, →T=532N· m

b) Bar with rectangular cross section

$\tau_{max}=T/c_1ab^2$, →T=414N· m

c) Square tube : $\tau_{max}=T/2tA$, →T=555N· m

# 5.2 Unsymmertic bending of beams and the principal centroidal axes of the cross section

- Area moments and products of inertia ---- definitions
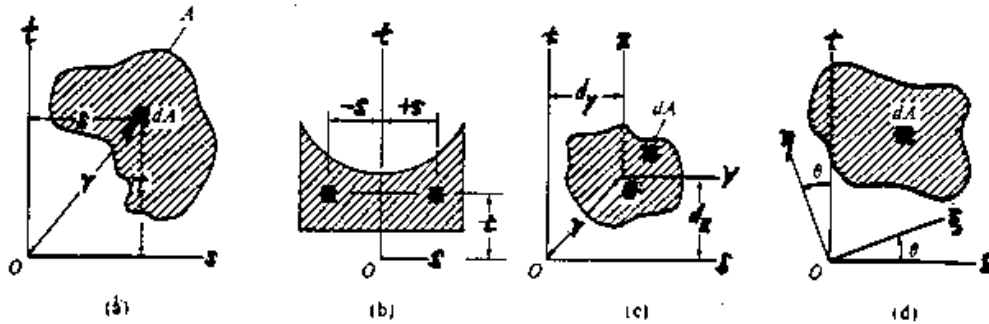


**FIGURE 9.1.1.** (a) Arbitrary plane area A with differential element dA. (b) Area symmetric about the t axis. (c) Centroidal axes yz and distances $d_y$ and dz used in parallel axis theoremes. (d) Orthogonal axis systems st and $\xi\eta$. having common origin but different orientation.

(1)    Moment of inertia about s and t axes

$$I_s = \int_{area} t^2 dA, \quad I_t = \int_{aera} s^2 dA \qquad (I_s>0, I_t>0)$$

(2)    Polar moment of inertia about point o ($I_o>0$)

$$I_o = \int_{aera} r^2 dA = \int_{aera} (t^2 + s^2) dA = I_s + I_t$$

(3)    Product of inertia about t and s axes

$$I_{st} = \int_{aera} st\, dA \qquad (I_{st} \text{ can be positive, negative or zero. If s}$$

or t is a symmetry axis of area, then $I_{st}=0$ )

- Principal Axes

  For a given area A and a given origin O, there are two axes s' ,t' about which the moments of inertia are, respectively, greater and less than about any other axes through origin O. There two axes are called <u>principal</u> <u>axes</u>, the corresponding moments of inertia are <u>principal moments of inertia (where</u> $\underline{I_{s't'}=0)}.$

N.A.

$$\sigma_x = \frac{M_y z}{I_y}$$

N.A.
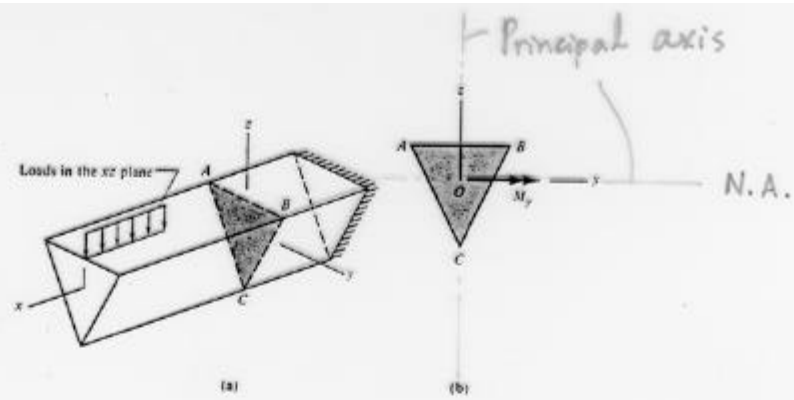
**FIGURE 9.2.1.** Elastic flexural stress in a beam where the $xz$ plane is a plane of symmetry. The loads produce moment $M_y$, which acts on material behind the $yz$ plane. Vector $M_y$ is parallel to the $y$ axis, which is a principal axis of the cross-sectional area. Point $O$ is the centroid of the cross section.

N.A.

Neutral axis

N.A.
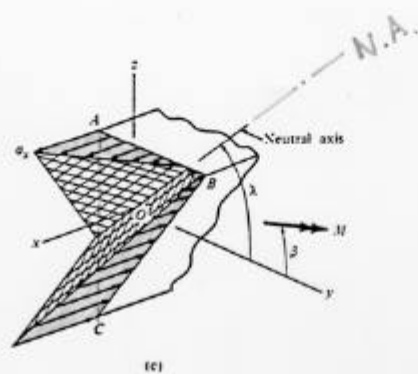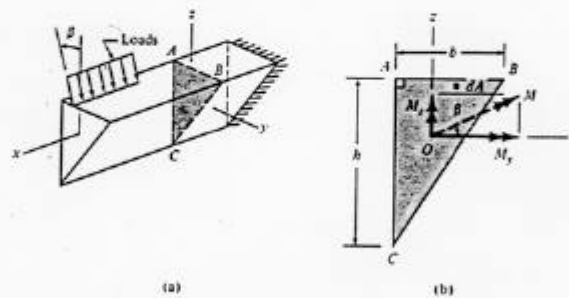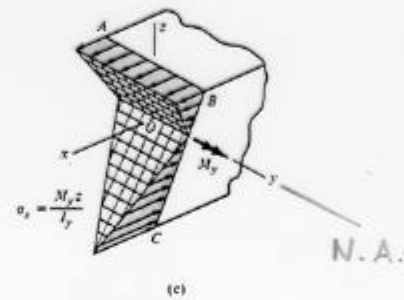
- How to decide the position of Principal Centroidal axes

(1) Find the centroid of the cross section and select axes s and t

(2) Calculate the $I_s$ and $I_t$, $I_{st}$

(3) Calculate the angle between Principal Centroidal axes and axes t,s by

$$\tan 2\boldsymbol{q}_m = \frac{2I_{st}}{I_t - I_s}$$

(4) Calculate the Principal moment of inertia

$$I_{max(min)} = \frac{I_s + I_t}{2} \pm \sqrt{\left(\frac{I_s - I_t}{2}\right)^2 + I_{st}^{\;2}}$$

If $I_{max} = I_{min}$, then all axes are principal

(as in the case of centroidal axes of circular cross section)

- There are exact analogy between principal stress(Mohr's circle) analysis and the principal moment of inertia

$I_{max}$ , $I_{min}$ , $I_{st}$

↓ ↓ ↓

$\sigma_{max}$ , $\sigma_{min}$ , $\tau_{xy}$

- The **precise condition** under which the **neutral axis** of a cross section of arbitrary shape **coincides** with the **axis of the couple M**

(1) Assume both couple vector M and the neutral axis of the cross section to be directed along Z axis



Fig. 4.51
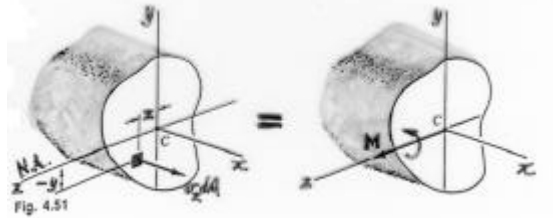
(2) The elementary force $\sigma_x dA$ must satisfy

- $\int \boldsymbol{s}_x dA = 0 \Rightarrow \int -\boldsymbol{s}_m \dfrac{y}{c} dA = -\dfrac{\boldsymbol{s}_m}{c} \int y dA = 0 \Rightarrow$

  $\int y dA = 0 \Rightarrow$ require neutral axis be a centroidal axis

- $\int (-y \boldsymbol{s}_x dA) = M \Rightarrow \int (-y)(-\dfrac{\boldsymbol{s}_m}{c} y) dA = M \Rightarrow$

  $\boldsymbol{s}_x = -\dfrac{My}{I}$    --------Elastic flexure formula

- $\int z \boldsymbol{s}_x dA = 0 \Rightarrow \int z(-\dfrac{\boldsymbol{s}_m y}{c}) dA = 0 \Rightarrow$

  $\int yz dA = 0 \Rightarrow$ Axes y and z are principal centroidal axes

  product of inertia $I_{yz}$

**CONDITION:If, and only if, the couple vector M is directed along one of the principal centroidal axes of the cross section**

- **Sample problems:**

(1) In all the 4 cases, the couple vector M is directed along one of the principal centroidal axes so the beutral axes are all coincident with the axes of the couples, even though some couples are not in a plane of symmetry or even though some cross section have no axes of symmetry.
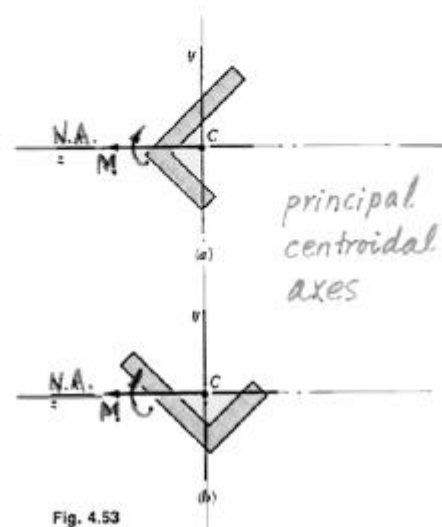


principal. centroidal axes
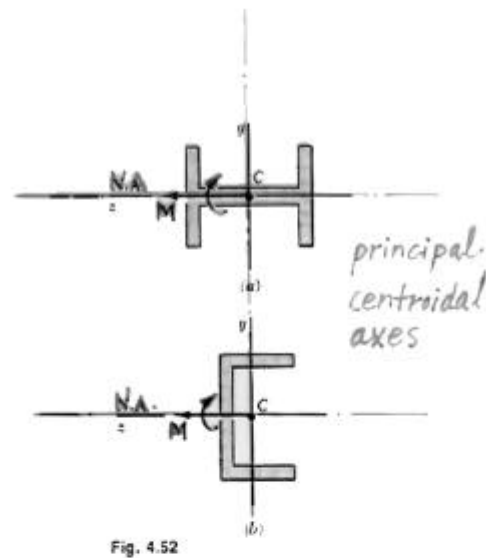
Fig. 4.52

(2) In all these cases, we can use the formula

$$S_x = -\frac{My}{I}$$

to calculate stresses directly.
I refers to the principal centroidal axes.



principal centroidal axes

Fig. 4.53

- The above formula can also be used to calculate the stress in Fig 4.58 , once the principal centroidal axes of the cross section have been determined
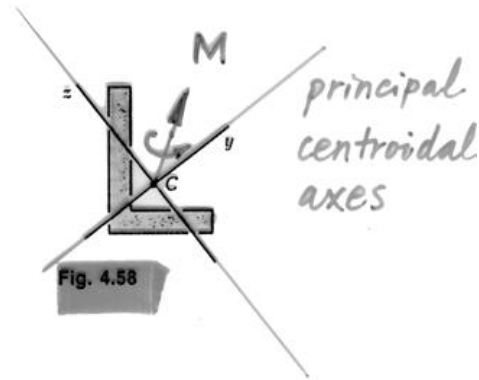


Fig. 4.58

- As we indicated earlier, the neutral axis of the cross section will not, in general, coincide with the axis of the bending couple, as shown in Fig.4.59 Since the normal stress at any point of the neutral axis is zero, we can determine the neutral axis by the following equation:
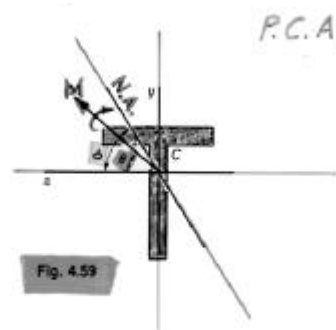


Fig. 4.59

$$s_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = 0$$

$$\Rightarrow y = (\frac{I_z}{I_y} \tan q) z$$

$$\tan q = \frac{M_y}{M_z} = \frac{M \sin q}{M \cos q}$$

a straight line of slope m=$(I_z/I_y)$tan$\theta$

$\phi>\theta$ when $I_z>I_y$ , $\phi<\theta$ when $I_z<I_y$

- Determination of stresses in unsymmetric bending ----
  **Principal of superposition**

(1) Resolving the vector M into vectors $M_y$ and $M_z$ along y and z axes
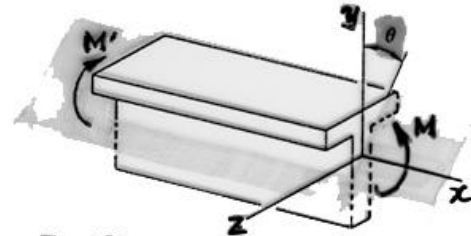
$M_z = M\cos\theta$, $M_y = M\sin\theta$

Fig. 4.54

(2) Since y and z axes are principal centroidal axes of the cross section, we can directly write:

$\sigma_x$ due to $M_z$ is

$$\sigma_x = -\frac{M_z y}{I_z}$$

$\sigma_z$ due to $M_y$ is

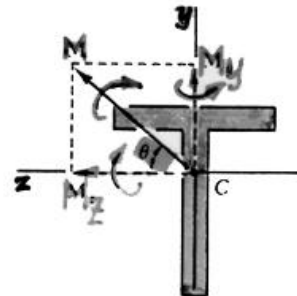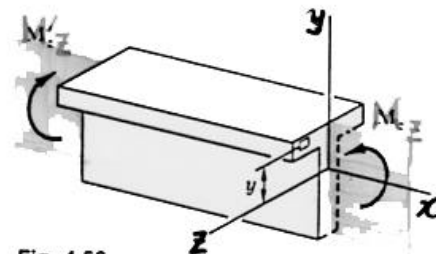$$\sigma_x = +\frac{M_y z}{I_y}$$

Fig. 4.55

Fig. 4.56

(3) The total stress is

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$
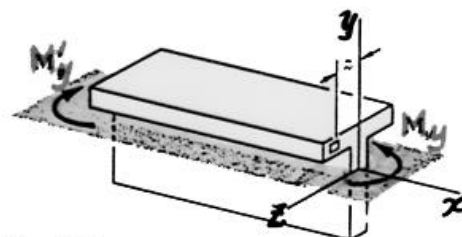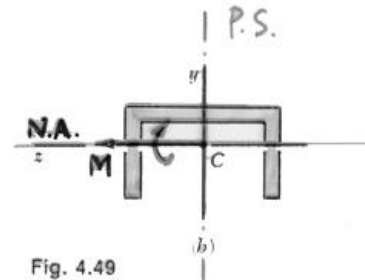
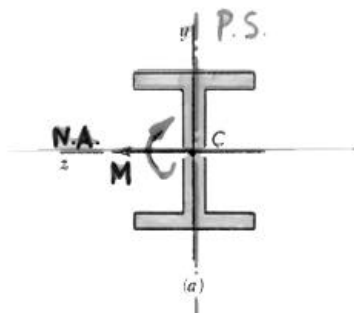The stress distribution in unsymmetric bending is linear.

Fig. 4.57

# 4.9 UNSYMMETRIC BENDING (MECH101)

**Symmetric bending**: Where the member possesses at least one plane of symmetry and is subjected to couples acting in that plane



Fig. 4.49

------ the neutral axis of the cross section is found to **coincides** with the axis of the couple

• **Unsymmetric bending**: Situations where couples do **not act in a plane of symmetry** of the member or the member does not possess any plane of symmetry



Fig. 4.50

*no plane of symmetry*
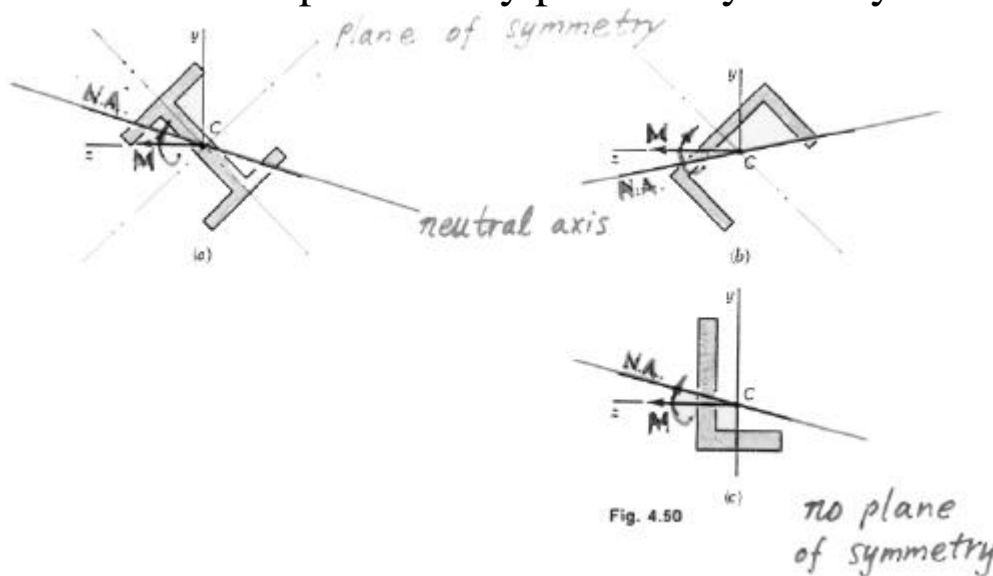
------ the neutral axis of the cross section usually **does not coincide** with the axis of the couple

# 4.10 GENERAL CASE OF ECCENTRIC AXIAL LOADING

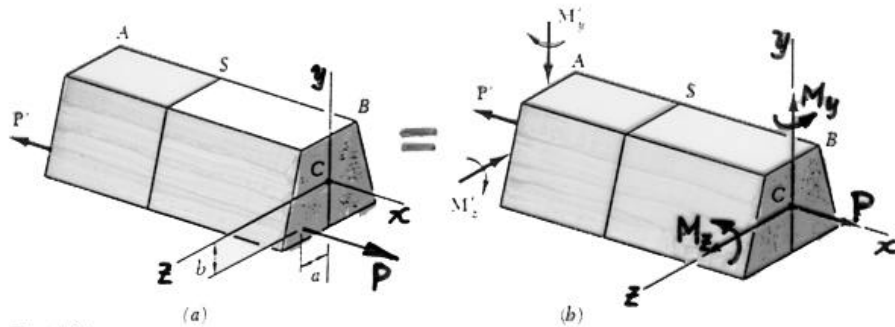- **More general cases:** axial load is not in plane of symmetry



Fig. 4.64

(1') First find the principal centroidal axes of the cross section

(1) Saint-Venant's principal----replace the original loading by the statically equivalent loading

(2) Principal of superposition

$$s_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

(be careful to determine the sign of each of the three terms)

S 250 × 38

120 mm

35 mm

C

P



$y$

254 mm

C

$x$

118 mm



$y$

B

A

$M_y$

$x$

$M_x$

C

P

E

D
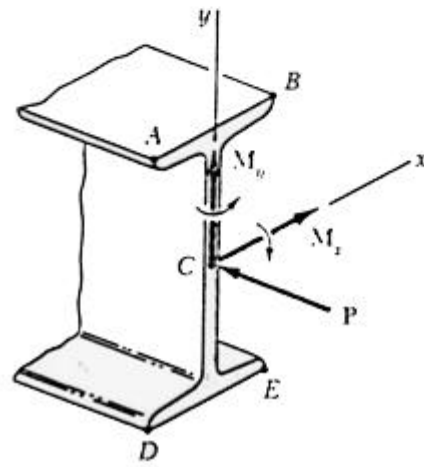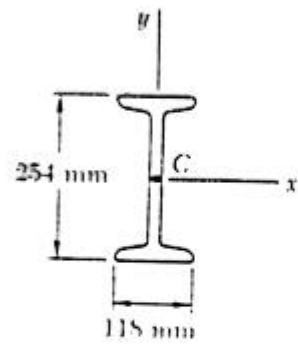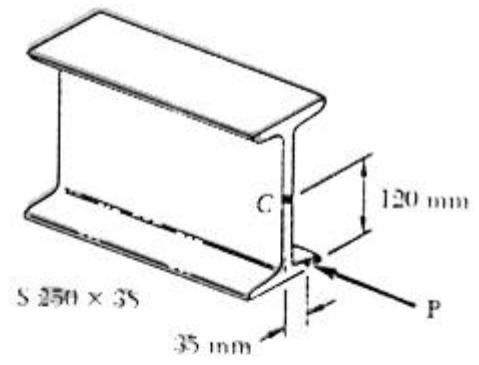
# 5.3 Unsymmetric loading of thin-walled members, shear center (MECH 101,pp 320-327)

- Stress formulas for transverse loading on cross section of thin-walled member with a vertical plane of symmetry

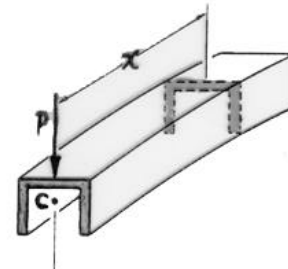$$s_x = -\frac{My}{I}$$

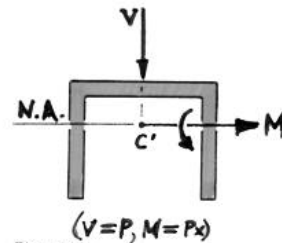$$t_{ave} = \frac{VQ}{It}$$

Fig. 5.51

Fig. 5.52
(V=P, M=Px)

- Transverse loading on thin-walled members which do not possess a vertical plane of symmetry(**bend and twist**)
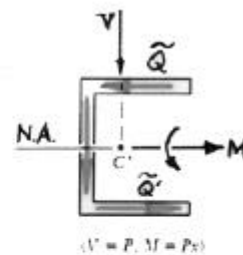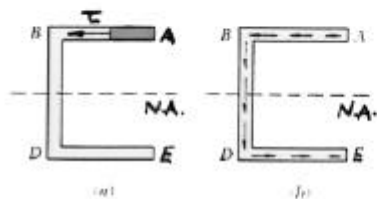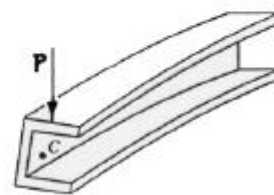
Fig. 5.53
(V = P, M = Px)
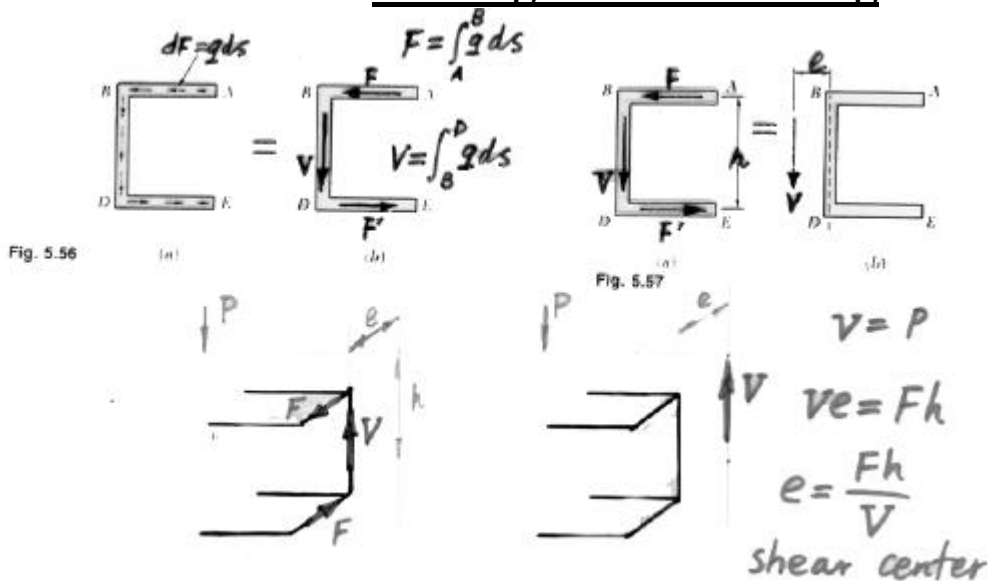
Fig. 5.55

Fig. 5.54

- The condition for **bending without twisting**



Fig. 5.56

$dF = q ds$     $F = \int_A^B q\, ds$

$V = \int_B^D q\, ds$

Fig. 5.57

$V = P$

$Ve = Fh$

$e = \dfrac{Fh}{V}$

shear center

- Calculation of the position of **Shear Center**
- When the force P is applied at a distance e to the left of the center line of the web BD, the member bends in a vertical plane without twisting



Fig. 5.58

- The point O where the line of action of P intersects the axis of symmetry of the end section is called shear center.
- Any oblique load P is applied through the shear center, the member will be also free of twisting, since it can be decompose into $P_y$ and $P_x$.



Fig. 5.59
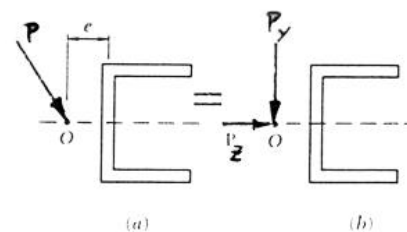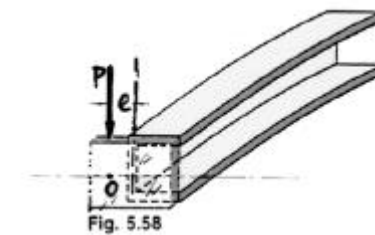
- ## Example 5.04

Determine the shear center O of a channel section of uniform thickness, b=100mm, h=150mm, t=3mm

(1) Assume that the member does not twist, the shear flow q in flange AB at a distance from A

$$q = \frac{VQ}{I} = \frac{Vsth}{2I}$$

Fig. 5.60

(2) The magnitude of the horizontal shear force F on flange AB

$$F = \int_0^b q\,ds = \frac{Vthb^2}{4I}$$

Fig. 5.61

(3) The distance e from the center line of web BD to the center O is

$$e = \frac{Fh}{V} = \frac{Vthb^2}{4I}\frac{h}{V} = \frac{th^2b^2}{4I}$$

where

$$I = I_{web} + 2I_{flange} = \frac{1}{12}th^3 + 2\left[\frac{1}{12}bt^3 + bt\left(\frac{h}{2}\right)^2\right]$$

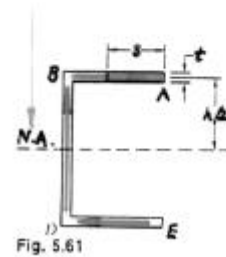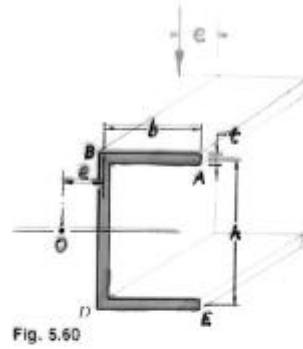$$\approx \frac{1}{12}th^2(6b + h) \quad \text{(neglecting } t^3\text{)}$$

So finally

$$e = \frac{3b^2}{6bth} = \frac{b}{2 + \dfrac{h}{3b}}$$ , we can see that e does not depend

on t,depends on the ratio h/3b,

e=0~b/2 ,For the given size here, $e = \dfrac{100mm}{2 + 0.5} = 40mm$

- ## **Example 5.05**

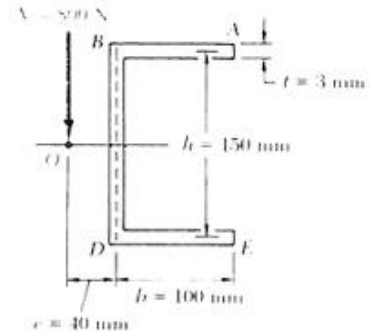Determine the distribution of the shearing stresses caused by 800N vertical shear V at the shear center O



Fig. 5.62

(1) Since V is applied at the shear center, there is no torsion. The shearing stress in the flange AB can be calculated as



Fig. 5.63

$$t = \frac{q}{t} = \frac{VQ}{It} = \frac{Vh}{2I}s$$

So the shear distribution is a linear function of s along AB.
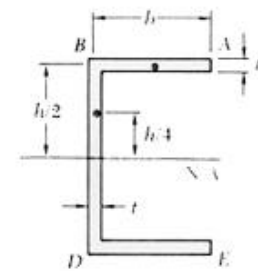
$$t_B = \frac{6Vb}{th(6bth)} = 1.422 Mpa$$

(2) The distribution of the shearing stress in the web is parabolic,

$$Q = \frac{1}{8}ht(4bth) \quad \text{at N.A.}$$

So:

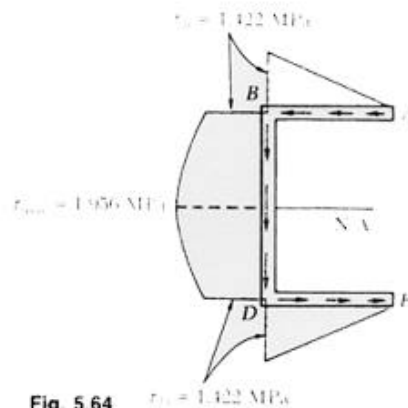$$t_{max} = \frac{VQ}{It} = \frac{3V(4bth)}{2th(6bth)} = 1.956 Mpa$$
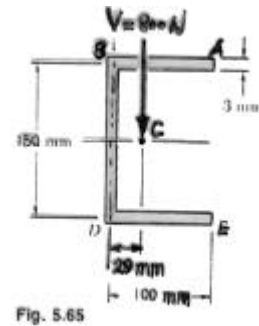


Fig. 5.64

- ## **Example 5.06**

For the channel section of the beam, determine the maximum shearing stress caused by V=800N applied at the centroid of the section(neglecting stress connection)



Fig. 5.65

(1) Determine the shear center, e=40mm

(2) Replace the shear V at the centroid by an equivalent force-couple system at the shear center O: the couple (torque) T=V (oc)=55.2N• m

(3) Suppose the shear stress due to V (at e=40mm, bending without twisting) with the shear stress produced by the torque T, we get $\tau_{max}$ in the section:

$t_{max}$ (bending due to V) =1.956Mpa (lost example)

$$t_{max} \text{ (torsion)} = \frac{T}{c_1 ab^2} = 52.8 Mpa$$

$$t_{max} = 1.956 + 52.8 = 54.8 Mpa$$



Fig. 5.66
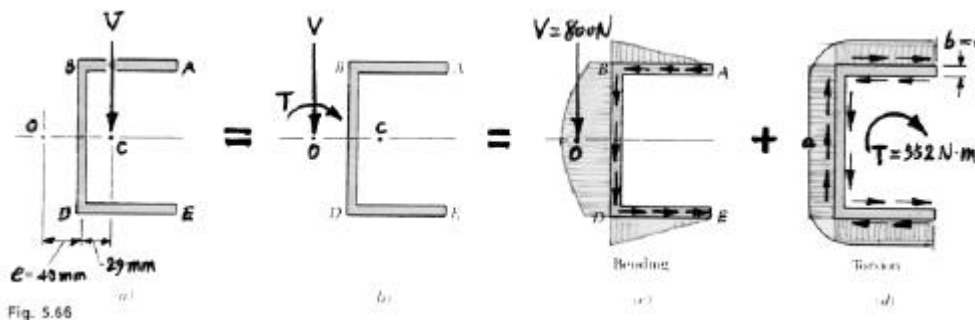
(4) From this example, we can see that the contribution to $\tau$ due to the torsion is significant !

- Thin-walled members with no plane of symmetry, position of the shear center

(1) If the load is perpendicular to one of the principal centroidal axes CZ of the cross section, then the member will not be twisted if the load pass through the corner O along vertical line.
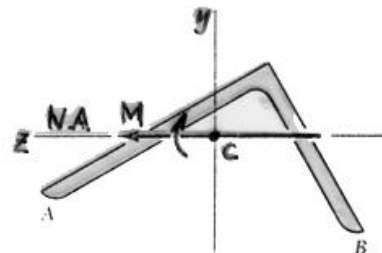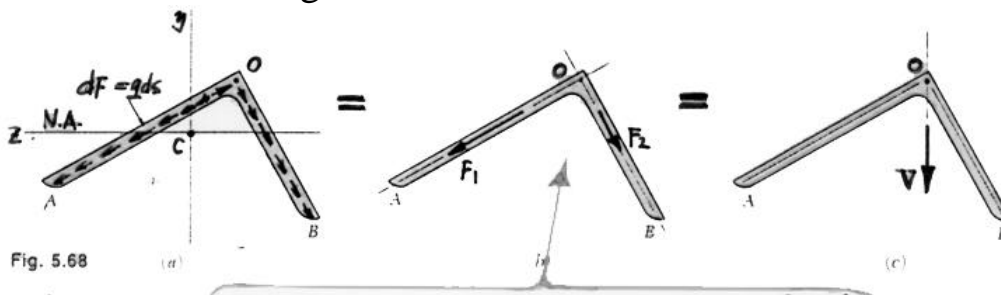


Fig. 5.67



Fig. 5.68

(2) The corner O is the shear center of the cross-section.
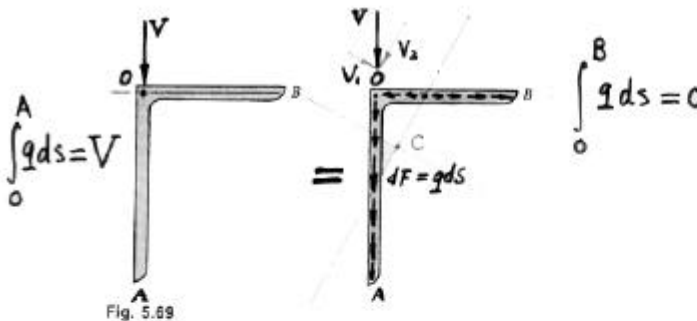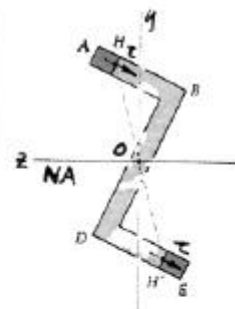(3) For the Z-shape cross section, the shear center is at O.



Fig. 5.69

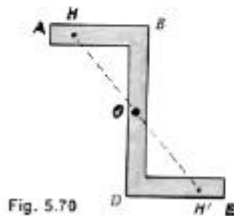$$\int_O^A q\,ds = V$$

$$\int_O^B q\,ds = 0$$

Fig. 5.71

Fig. 5.70
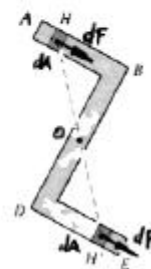
Fig. 5.72

- ## Example 5.8

Determine the distribution of shearing stresses in the thin-walled angle shape DE of uniform thickness t for the loading shown.

(1) Shear center: since P pass through O, there is no twisting.

(2) Principal centroidal axes through point C (Centroid)

(3) Resolve shear V(=P) into $V_y$, $V_z$, which are parallel to the principal axes, and calculate the corresponding shear stress, respectively, by formulas

$$t_1 = \frac{V_{y'} - Q}{I_{z'}t}\ ,$$

$$t_2 = \frac{V_{z'} - Q}{I_{y'}t}$$

(4) Obtain the total shear stress τ by

$$t = t_1 + t_2$$