# CHAPTER FOUR

## ELASTIC FOUNDATIONS

| * | Bending of beams on elastic foundations and solutions |
| ** | Solution by superposition and Contact stress problems |

## 4.1 Introduction and Foundation Models ---- Winkler Foundation

- Concept of Elastic Foundations and the Effect of the Foundation on the Beam (a kind of contact)

* Not to study the stresses in the foundation itself.
Two Analytical Models on Elastic Foundation

(1) Model 1 ---- Winkler Model ---- a linear force-deflection relationship is presumed

FIGURE 5.1.1. Deflections of foundation models under uniform pressure. No beam is present.

A linear relationship between the force on the foundation (pressure $p$) and the deflection $w$ is assumed:

$$ p = k_0 w \quad k_0 \text{ is the foundation modulus (unit: N/m}^2/\text{m}) $$

For beams with width $b$, we use

$$ p = kw = k_0 bw, \text{ unit of } k: \text{N/m/m} $$

** An Important restriction of the model: the contact is never broken between beam and foundation

(2) Model 2 ---- Elastic solid Foundation ---- More realistic but bore complicated (not used here)
4.2 Governing Equations For Uniform Straight Beams on Elastic Foundations

- Governing Equations

(1) In **Usual Beam Theory** (MECH 101)

\[
\begin{align*}
\frac{dV}{dx} & = -q, \quad \frac{dM}{dx} = V \rightarrow \frac{d^2 M}{dx^2} = -q \\
M & = -EI \frac{d^2 w}{dx^2} \rightarrow EI \frac{d^4 w}{dx^4} = q
\end{align*}
\]

(2) Beam Theory on **Winkler Foundation**

\[
\begin{align*}
\frac{dV}{dx} & = -q + kw, \quad \frac{dM}{dx} = V \rightarrow \frac{d^2 M}{dx^2} = kw - q \\
M & = -EI \frac{d^2 w}{dx^2} \rightarrow EI \frac{d^4 w}{dx^4} + kw = q
\end{align*}
\]

**FIGURE 5.2.1.** (a) Arbitrary loading on an elastically supported beam. (b) Reaction kw of a Winkler foundation. The curve \( w = w(x) \) is the deflected shape of the beam. (c) Forces that act on a differential element of the beam.
• Solution of the Equation

The governing equation for a uniform beam on Winkler foundation:

\[
EI \frac{d^4 w}{dx^4} + kw = q
\]

By introducing a parameter \( \beta \) (unit L\(^{-1}\))

\[
\beta = \left[ \frac{k}{4 EI} \right]^{\frac{1}{4}}
\]

The solution of the governing equation can be written as

\[
w = e^{\beta x} (C_1 \sin \beta x + C_2 \cos \beta x) + e^{-\beta x} (C_3 \sin \beta x + C_4 \cos \beta x) + w(q)
\]

Particular solution related with \( q \),

\( w(q) = 0 \) when \( q = 0 \)

\( C_1, C_2, C_3, C_4 \) are constants of integration, which are determined by B.C. When \( w(x) \) is known, \( V, M, \theta, \sigma \) etc can be calculated by the relevant formulas.

For the convenience, the following symbols are defined:

\[
A_{\beta x} = e^{-\beta x} (\cos \beta x + \sin \beta x), B_{\beta x} = e^{-\beta x} \sin \beta x
\]

\[
C_{\beta x} = e^{-\beta x} (\cos \beta x - \sin \beta x), D_{\beta x} = e^{-\beta x} \cos \beta x
\]

These quantities are related by certain derivatives, and the value of the above quantities are listed in the table.
TABLE 5.2.1 Selected Values of Terms Defined by Eqs. 5.2.7.

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4.3 Semi-infinite and Infinite Beams with Concentrated Loads

- Semi-infinite beams with concentrated load

\[ M(x) = \frac{2\beta P_o}{k} D_{\beta x} - \frac{2\beta^2 M_o}{k} C_{\beta x}, \]

\[ \theta = \frac{dw}{dx} = -\frac{2\beta}{k} A_{\beta x} + \frac{4\beta^3 M_o}{k} D_{\beta x}, \]

\[ w(x) = \frac{2\beta P_o}{k} D_{\beta x} - \frac{2\beta^2 M_o}{k} C_{\beta x}, \]

**FIGURE 5.3.1.** (a) Concentrated loads \( P_o \) and \( M_o \) at the end of a semi-infinite beam on a Winkler foundations.  (b) End deflection \( w_o \) and end rotation \( \theta_o = (dw/dx)_{x=0} \) both shown in the positive sense.

- Two kind of boundary conditions:

  (1) prescribe \( P_o \) and \( M_o \) at \( x = 0 \)

  (2) prescribe \( w_o \) and \( \theta_o \) at \( x = 0 \)

- For Boundary condition (1)

  Let \( w(q) = 0 \) in the general expression of solution.  Since \( w = 0 \) at \( x \to \infty \), we must have \( C_1 = C_2 = 0 \).  The other two boundary conditions determine \( C_3, C_4 \).

  \[ M|_{x=0} = -EI \frac{d^2w}{dx^2} \bigg|_{x=0} = M_o \rightarrow C_3 = \frac{2\beta^2 M_o}{k} \]

  \[ V|_{x=0} = -EI \frac{d^3w}{dx^3} \bigg|_{x=0} = 0 \rightarrow C_4 = \frac{2\beta P_o}{k} - \frac{2\beta^2 M_o}{k} \]

  So finally we have:
All $M(x)$, $V(x)$, $w(x)$, $\theta(x)$ are damped sine and cosine waves.

**Example**

A semi-infinite steel bar ($E = 200\text{GPa}$) has a square cross section ($b = h = 80\text{mm}$) and rests on a Winkler foundation of modulus $k_0 = 0.25 \text{N/mm}^2/\text{mm}$. A downward force of $50\text{kN}$ is applied to the end. Find the maximum and minimum deflections and their locations. Also find max. flexural stress and its location.

(1) Necessary constants are:

$$EI = 200000 \frac{80^4}{12} = 6.827 \times 10^{11} \text{N} \cdot \text{mm}^2, k = 80k_0 = \frac{20N}{\text{mm} \cdot \text{mm}}.$$  

$$\beta = \left[\frac{k}{4EI}\right]^{\frac{1}{2}} = 0.001645/\text{mm}$$

(2) The displacement $w(x) = 2\beta P_0 D_{\beta x} / k$

$$w_{\text{max}} = w|_{x=0} = w_o = \frac{2\beta P_o}{k} = 8.225\text{mm}$$

The min. deflection occurs at the smallest distance for which $\theta = 0$. From $\theta = -\frac{2\beta^2 P_o}{k} A_{\beta x}$, we find $A_{\beta x} = 0$ at $\beta x = 3\pi/4$ or $x = 1432\text{mm}$, corresponding $D_{\beta x} = -0.0670$, so $w_{\text{min}} = \frac{2\beta P_o D_{\beta x}}{k} = -0.551\text{mm}$

(This upward deflection reminds us our assumption on the beam – foundation connection)
(3) Bending moment is $M = -P_0 B_{\beta x} / \beta$, from the table we find that $B_{\beta x}$ has largest value at $\beta x = \pi/4$, the corresponding $B_{\beta x} = 0.3234$, so $M_{\text{min}} = -9.8 \times 10^6$ Nmm

$\sigma_{\text{max}} = \frac{Mc}{I} = 115\text{MPa}$ appears on top of the beam at $x = \pi/4\beta = 477\text{mm}$.

- **Infinite beams with concentrated load**

(1) Concentrated force ---- by using previous solution ---- equivalent to:

![Diagram](image.png)

**FIGURE 5.4.1.** (a) Concentrated load $P_0$ at $x = 0$ on a uniform infinite beam that rests on a Winkler foundation. (b-e) Curves for deflections, rotation, bending moment, and transverse shear force in the beam. These curves are proportional to $A_{\beta x}$, $B_{\beta x}$, $C_{\beta x}$, $D_{\beta x}$, respectively.
By using the solution for semi-infinite beam under concentrated load, we have:

at \( x = 0 \), \( \theta = -\frac{2\beta^3 \left( \frac{P_o}{2} \right)}{k} + \frac{4\beta^2 M_o}{k} = 0 \), \( M_o = \frac{P_o}{4\beta} \) due to symmetry (mirror at \( x = 0 \)), we have \( V = 0 \) at \( x = 0 \).

Substituting \( P_o/2 \) and \( M_o = P_o/4\beta \) in the previous solution (semi-infinite beam under concentrated force and moment at the end), we obtain the solution for infinite beam here:

\[
\begin{align*}
  w &= \frac{\beta P_o}{2k} A_{\beta x} ; \\
  \theta &= \frac{d}{dx} \left( \frac{\beta^2 P_o}{k} B_{\beta x} \right) ; \\
  M &= \frac{P_o}{4\beta} C_{\beta x} ; \\
  V &= \frac{P_o}{2} D_{\beta x} .
\end{align*}
\]

**Notes:** In these solutions, \( x \) should be \( x \geq 0 \), for \( x < 0 \), the \( w(x), M(x), \theta(x) \) and \( V(x) \) must be obtained from the symmetry and antisymmetry conditions: \( w(x) = w(-x), \theta(x) = -\theta(-x), M(x) = M(-x), V(x) = -V(-x) \).
(2) Concentrated moment ---- by using previous solution ----
equivalent to:

FIGURE 5.4.2. (a) Concentrated moment $M_o$ at $x = 0$ on a uniform infinite beam that rests on a Winkler foundation. (b-e) Curves for deflection, rotation, bending, moment, and transverse shear force in the beam. These curves are proportional to $B_{\beta x}$, $C_{\beta x}$, $D_{\beta x}$, and $B_{\beta x}$, respectively.

(1) Deformation analysis: Deflections are antisymmetric with respect to the origin; so $w|_{x=0} = 0$. Bending moment $M|_{x=0^+} = \frac{M_o}{2}$, $M|_{x=0^-} = -\frac{M_o}{2}$. Substituting into the expression $w(x)$ for semi-infinite beam with concentrated load,

$$w|_{x=0} = 0 = \frac{2\beta P_o}{k} - \frac{2\beta (M_o/2)}{k} \rightarrow P_o = \frac{M_o \beta}{2}$$
(2) Then substituting \( P_o = \frac{M_o \beta}{2}, M = \frac{M_o}{2} \) into basic solution, we have

\[
\begin{align*}
    w(x) &= \frac{\beta^2 M_o}{k} B_{\beta x}, \\
    \theta &= \frac{dw}{dx} = \frac{\beta^3 M_o}{k} C_{\beta x} \\
    M(x) &= \frac{M_o}{2} D_{\beta x}, V = -\frac{\beta M_o}{2} A_{\beta x},
\end{align*}
\]

The solutions for the left half of the beam must be obtained from the following symmetry and antisymmetry conditions:

\[
\begin{align*}
    w(x) &= -w(x); \theta(x) = \theta(-x); M(x) = -M(-x); \\
    V(x) &= V(-x).
\end{align*}
\]
Example: \[ P = 18\text{kN} \]

A infinite beam rest on equally spaced linear coil springs, located every 1.1m along the beam. A concentrated load of 18kN is applied to the beam, over one of the springs. EI of the beam is \( 441 \times 10^9 \text{Nmm}^2 \), \( K = 275 \text{N/mm} \) for each spring. Compute the largest spring force and largest bending moment in the beam.

(1) To “smear” the springs into a Winkler foundation:

force applied to the beam by a spring with deflection \( w \) is \( Kw \), so if the spring spacing is \( L \), the associated force in each span \( L \) is \( Kw \), then the hypothetical distributed force is therefore \( Kw / L \)

\[
\text{distributed force } \frac{kw}{L} = \frac{\text{distributed force } Kw}{L} \text{ of Winkler foundation by a series of springs}
\]

The “equivalent” Winkler foundation modulus is \( k = \frac{K}{L} \) and the

\[
\beta = \left[ \frac{k}{4EI} \right]^\frac{1}{4} = 6.136 \times 10^{-4} /\text{mm}
\]
(2) According to the previous solution for infinite beam with concentrated load $P$, we have

$$w_{\text{max}} = w|_{x=0} = \frac{\beta P_o}{2k} A_{\beta x} = 22.1\text{mm},$$

the maximum spring force is $F_{\text{max}} = K w_{\text{max}} = 6075\text{N}$.

$$M_{\text{max}} = M|_{x=0} = \frac{P_o}{4\beta} C_{\beta x} = 7.33\text{kN \cdot m}$$

(3) If the beam length is finite with several springs, then the problem can be solved as static indeterminate beam.
Example:

FIGURE 5.4.3. (a) Equal concentrated loads on an elastically supported beam. (b-c) Resulting deflection and bending moment. Dashed lines represent results of individual loads. Solid lines are superposed results. (d-f) Coordinate systems used to solve the problem by superposition.

An infinite beam on a Winkler foundation has the following properties: 

\[
EI = 441 \times 10^9 N \cdot mm^2, k = 0.25 N / mm / mm \\
\beta = 6.136 \times 10^{-4} / mm
\]

Two concentrated loads, 18kN each and 2.6m apart, are applied to the beam. Determine \( w_{\text{max}} \) and \( M_{\text{max}} \).

Principal of superposition: total \( w \) and \( M \) are

\[
w = w_1 + w_2; M = M_1 + M_2
\]
We find that \( w_{\text{max}} \) is at point B, \( M_{\text{max}} \) is at A and C. The resultant \( w \) is larger than a single load, but resultant \( M \) is a little smaller than the case of a single load.

4.4 Semi-infinite and Infinite Beams with Distributed Loads, Short Beams

- **Semi-infinite beam** with distributed load over the entire span

\[
\begin{align*}
\text{(I)} & & \text{(II)} \\
M = 0 & & M = 0 \\
w = 0 & & w = 0
\end{align*}
\]

\[ k \frac{q_o}{x} \]

\[ \text{due to } q_o \]

**FIGURE 5.5.1.** (a) Semi-infinite beam on a Winkler foundation, loaded by end force \( P_o \), end moment \( M_o \), and a uniformly distributed load \( q_o \) over the entire beam. (b) Deflected shape of the beam if simply supported and loaded by \( q_o \) only.

(I) Analysis: since \( q_o \) is added to the entire beam, we begin with the general solution. At large \( x \), the beam does not bend. There the load is carried by the foundation uniformly with deflection \( q_o / k \). So in the general solution, we have \( C_1 = C_2 = 0 \) and \( w(q) = q_o / k \), and

\[
w = C_3 B_{\beta x} + C_4 D_{\beta x} + \frac{q_o}{k} \]

\[ \text{due to } q_o \]
The boundary condition at $x = 0$ leads to

$$M|_{x=0} = M_0 \rightarrow C_3 = \frac{2\beta^2 M_0}{k}, V|_{x=0} = -P_0 \rightarrow C_4 = \frac{2\beta P_0}{k} - \frac{2\beta^2 M_0}{k}$$

The solutions are finally,

$$w = \frac{2\beta P_0}{k} D_{\beta x} - \frac{2\beta^2 M_0}{k} C_{\beta x} + \frac{q_o}{k} \quad \text{(due to } q_o)$$

$$\theta = -\frac{2\beta^2 P_0}{k} A_{\beta x} + \frac{4\beta^3 M_0}{k} D_{\beta x}, M = ..., V = ...$$

(II) In this case, the boundary conditions are $M|_{x=0} = 0$,

$$w|_{x=0} = 0 \rightarrow C_3, C_4 \rightarrow w \rightarrow \text{support reaction at } x = 0 \text{ to be } \frac{q_o}{2\beta}.$$

- **Infinite beam with distributed load over a length $L$**

\[\text{FIGURE 5.5.2. Uniformly distributed load } q_o, \text{ over a length } L = a + b \text{ of an infinite beam on a Winkler foundation.}\]

(1) Method: Principal of superposition
(2) Basic solution: infinite beam under concentrated force $P$

$$w = \frac{\beta P_o}{2k} A_{\beta x}$$

(3) The deflection at $Q$ due to load $q_o dx$ at $O$ is

$$dw_Q = \frac{\beta q_o dx}{2k} A_{\beta x}$$

(4) The total deflection at $Q$ is

$$w_Q = \frac{\beta q_o}{2k} \left[ \int_0^a A_{\beta x} dx + \int_0^b A_{\beta x} dx \right] = -\frac{q_o}{2k} \left[ D_{\beta x} \bigg|_0^a + D_{\beta x} \bigg|_0^b \right]$$

$$w_Q = \frac{q_o}{2k} \left( 2 - D_{\beta a} - D_{\beta b} \right)$$

(5) By the same integration, we get the total $M$ at $Q$

$$M_q = \frac{q_o}{4\beta} \left( B_{\beta a} + B_{\beta b} \right)$$

and

$$\theta_Q = \frac{\beta q_o}{2k} (A_{\beta a} - A_{\beta b}) V_Q = \frac{q_o}{4\beta} \left( C_{\beta a} - C_{\beta b} \right)$$
It is helpful to identify three cases:

(I) $\beta L$ is small (or $\beta$ is small), $L$ is small: The deflection and bending moment are greatest at the middle of the span $L$, the corresponding condition is that $\beta L \leq \pi$.

(II) $\beta L$ is large: (1) deflection is constant in the center portion $w = q_o / k$, and bending moment is zero except in the neighborhood of the ends of the loaded zone.

(III) Intermediate values of $\beta L$. $\pi < \beta L$.

**FIGURE 5.5.3** Deflection and bending moment in uniform and uniformly loaded infinite beams on a Winkler foundation.
Short beams on a Winkler foundation

**FIGURE 5.6.1.** (a) Centrally loaded beam of finite length on a Winkler foundation. (b) End deflection \( w_{\text{end}} \) at \( x = \pm L / 2 \), as a fraction of center deflection \( w_0 \), versus \( \beta L \). Also, the ratio of \( w_0 \) for a finite beam to \( w_0 \) for an infinitely long beam.

1. Four boundary conditions: At \( x = 0 \), \( \theta = 0 \) and \( V = \frac{P}{2} \)

\[
\text{at} \quad x = \frac{L}{2}, \frac{L}{2}, \quad M = V = 0
\]

2. Get four constants \( C_1, C_2, C_3, C_4 \), the results are known and are tabulated for several cases.

3. Also 3 cases can be classified: (a) short beams; (b) intermediate beams; (c) long beams. The ratio of center deflection changes with the length of the beam. The ratio of end deflection to center deflection is also plotted in the figure.
4.5 Contact Stress ---- Problem and Solutions

- Features of the contact problem

  (1) The area of contact between bodies grows as load increases

  (2) In the contact stress problem, stresses remain finite

- The pioneer work by Hertz in 1881

- Basic assumption:

  (1) The contacting bodies are linearly elastic, homogeneous, isotropic, and contacting zone is relatively small.

  (2) Friction is taken as zero → contact pressure is normal to the contact area.
Solution for two contacting spheres

**FIGURE 5.8.1.** Radius \( a \) of the contact area and peak contact pressure \( p_o \) for the cases of (a) two spheres of equal radius, and (b) a sphere on a half-space (which amounts to a sphere of infinite radius). Poisson’s ratio is taken as \( \nu = 0.3 \).

Contact area: circle of radius \( a = 1.109 \left( \frac{PR}{E} \right)^{1/3} \)

The maximum contact pressure \( p_o \)

\[
p_o = 0.388 \left( \frac{PE}{R^2} \right)^{1/3}
\]

when a sphere \( R_1 \) pressed into a spherical socket \( R_2 \), \( R_2 > R_1 \), the results are obtained by making \( R_2 \) negative!

Solution for two parallel contact cylinders of length \( L \) \((L \geq 10a)\)

(1) Contact area: long rectangle \( L \times 2a \)
(2) \[ a = 1.52 \sqrt{\frac{P}{LE}} \left( \frac{R_1 R_2}{R_1 + R_2} \right), p_o = 0.418 \sqrt{\frac{PE}{L}} \left( \frac{R_1 + R_2}{R_1 R_2} \right) \]

- Solution for two crossed cylinders \((R_1 = R_2)\)

(1) Contact area: circular

(2) \(a\) and \(p_o\) are obtained from equations in Fig. 5.8.1(b)

- Some discussions

(1) Contact pressure is not proportional to \(P\)

(2) Stress state in the center of the contact area between spheres \((x = y = z = 0)\)

\[ \sigma_x = \sigma_y = -\frac{1 + 2\nu}{2} p_o, \sigma_z = -p_o \]

**FIGURE 5.8.2.** (a) Circular contact area between two spheres. Contact pressure varies quadratically from a maximum of \(p_o\) at \(x = y = 0\). (b) Principal stresses and maximum shear stress along the axis of loads \(P\) in contacting spheres, for \(\nu = 0.3\). (c) Rectangular contact area between parallel cylinders.