CHAPTER THREE
SYMMETRIC BENDING OF CIRCLE PLATES

* Governing equations in beam and plate bending
** Solution by superposition

1.1 From Beam Bending to Plate Bending
1.2 Governing Equations For Symmetric Bending of Circular Plates,
   Some Typical Solutions
1.3 Solution by Superposition
1.1 FROM BEAM BENDING TO PLATE BENDING

- The similarity between the beam bending and plate bending (loads, deformation, basic assumption etc)

plate:  \( a \gg h, \ b \gg h \)

beam:  \( a \gg h, \ a \gg b \)
• The governing equations for beam bending

\[ M = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_x z b dz \]
\[ \sigma_x = \frac{Mz}{I} \]

\[ \frac{dQ}{dx} + q = 0 \]
\[ \frac{dM}{dx} - Q = 0 \]
\[ \frac{d^2M}{dx^2} + q = 0 \]
\[ w = w(x), u = -z \frac{dw}{dx}, \varepsilon_x = \frac{du}{dx} = -z \frac{d^2w}{dx^2} \]

\[ \sigma_x = E\varepsilon_x = -Ez \frac{d^2w}{dx^2} = \frac{Mz}{I} \]

\[ \frac{d^2 w}{dx^2} = -\frac{M}{EI} \]

(c) For constant EI, we further have

\[ \frac{d^4 w}{dx^4} = \frac{q}{EI} \]

given M, E, I (or q, E, I) and boundary conditions on w and its derivatives, find function w(x).
• **The results are based on the following assumptions** (they apply also to plates)

(1) An unloaded beam is straight, and b and t are constant. (for plate, it is flat and t is const.)

(2) The material is homogeneous, isotropic and linearly elastic.

(3) Loading: distributed lateral force q, shear force and bending moments on the beam ends (or plate edges). Forces are z-parallel and moment vectors are z-perpendicular

(4) The only stresses of significance are axial stress $\sigma_x$ in a beam, and $xy$-parallel stresses $\sigma_x$, $\sigma_y$, $\tau_{xy}$ in a plate. Each $z = \text{constant}$ layer is assumed to be uniaxially stressed in a beam and biaxially stressed in a plate.

(5) Transverse shear forces (Q in a beam, $Q_x$ and $Q_y$ in a plate) exist, but transverse shear strains are very small, so that line AB (CD) $\perp$ neutral axis (neutral surface) before and after loading.

\[ \frac{d^2w}{dx^2} \ll 1, \frac{d^2w}{dy^2} \ll 1 \]

(6) $w$ is small and $w < \frac{1}{10} t$, the z-direction deflection $w$ of the neutral surface describes all about the deformation state.
Plate Bending Equations in X-Y-Z Coordinates

\[ M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} zdz \]

\[ Q_y = \int_{-t/2}^{t/2} \tau_{yz} dz \]

\[ M_y = \int_{-t/2}^{t/2} \sigma_y zdz \]

\[ M = \int_{-t/2}^{t/2} \tau_{xz} dz \]
The displacement $w = w(x,y)$

The strain-displacement relations are

\[
\begin{align*}
  u &= -z \frac{\partial w}{\partial x}, \quad v = -z \frac{\partial w}{\partial y} \\
  \varepsilon_x &= \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = \frac{\partial v}{\partial y} = -z \frac{\partial^2 w}{\partial y^2} \\
  \gamma_{xy} &= -2z \frac{\partial^2 w}{\partial x \partial y} \\
  \sigma_x &= \frac{E}{1-\nu^2} \left( \varepsilon_x + \nu \varepsilon_y \right) = -\frac{Ez}{1-\nu^2} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\
  \sigma_y &= \frac{E}{1-\nu^2} \left( \varepsilon_y + \nu \varepsilon_x \right) = -\frac{Ez}{1-\nu^2} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\
  \tau_{xy} &= G \gamma_{xy} = -2Gz \frac{\partial^2 w}{\partial x \partial y} = -\frac{Ez}{1+\nu} \frac{\partial^2 w}{\partial x \partial y}
\end{align*}
\]

The equation for $w(x,y)$ is

\[
D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + q = 0
\]

When $w = w(x)$ → beam bending equation (before)
A simple case of plate bending ---- bending into a cylindrical surface

FIGURE 2.14.1. (a) Moments $M_x = -2c_1D$ bend the central portion of the plate to a cylindrical surface. (b) Deformation of a cross section if edges $y = \pm b/2$ are free. (c) Deformation of a cross section if $b$ and $t$ are of comparable size, as with a beam.
(1) In this case, the displacement field is \( w = w(x) = c_1 x^2 \) (\( c_1 \) is a constant)

(2) The associated bending and twisting moments are

\[
M_x = -2c_1D, \quad M_y = -2\nu c_1D, \quad M_{xy} = 0
\]

This means that if edge moment \( M_x \) is applied, moment \( M_y \) will automatically arise due to Poisson effect, so plate will be in a biaxial stress state.

\[
\varepsilon_y = -\nu \varepsilon_x = -\nu \left( \frac{\sigma_x}{E} \right) \quad \text{(in a beam)}
\]

\[
\varepsilon_y = 0 \quad \text{(in a plate)} \rightarrow \quad \varepsilon_y = 0 = \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right)
\]

\[\rightarrow \sigma_y = \nu \sigma_x \rightarrow M_y = \nu M_x\]

(3) If \( b < t \), we have a beam, \( \varepsilon_y = -\nu \varepsilon_x, \sigma_y = 0 \)
If \( b \gg t \), we have a plate, \( \varepsilon_y = 0 \rightarrow \sigma_y = \nu \sigma_x \\
\text{(in the central portion)} \quad M_y = \nu M_x\]
• Boundary conditions for thin-plates

**FIGURE 2.14.4.** Illustration of different boundary conditions. The left is (a) clamped, (b) simply supported, and (c) free.

(a) Clamped edge (neither deflects nor rotates)

\[ w = 0 \text{ and } \frac{\partial w}{\partial x} = 0 \text{ at } x = 0 \]

(b) Simply supported (no deflection and } \sigma_x = 0 \text{ at edge)

\[ w = 0 \text{ and } M_x = 0 \text{ at } x = 0 \]

(c) Free edge

\[ M_x = 0, \ M_{xy} = 0, \ Q_x = 0 \]
1.2 GOVERNING EQUATIONS FOR SYMMETRIC BENDING OF CIRCULAR PLATES, SOME TYPICAL SOLUTIONS

- The only unknown is the plate deflection \( w \) which depends on coordinates \( r \) only \((w = w(r))\) and determines all the forces, moments, stresses, strains and displacement in the plate.

(1) Axial Symmetry → \( \sigma_r , \sigma_\theta , \tau_r\theta \), \((\tau_r\theta = 0)\); \( M_r , M_\theta , Q_r \), \((Q_\theta = 0)\)

\[
\varepsilon_r = \frac{du}{dr} = -z \frac{d^2w}{dr^2} (u = u(r)) \\
\varepsilon_\theta = \frac{2\pi(r + u) - 2\pi r}{2\pi r} = \frac{u}{r} = -z \frac{dw}{dr} \\
\gamma_{r\theta} = 0 \quad (\tau_{r\theta} = 0)
\]
(2) Hooke’s law is expressed in terms of $w$, as follows

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu \varepsilon_\theta) = -\frac{Ez}{1-\nu^2}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr}\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu \varepsilon_r) = -\frac{Ez}{1-\nu^2}\left(\frac{1}{r} \frac{dw}{dr} + \nu \frac{d^2w}{dr^2}\right)$$

(3) Bending moment and shear force

$$M_r = -D \left(\frac{d^2w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr}\right), \quad D = \frac{Et^3}{12(1-\nu^2)}$$

$$M_\theta = -D \left(\frac{1}{r} \frac{dw}{dr} + \nu \frac{d^2w}{dr^2}\right)$$

$$Q_r = -\frac{1}{2\pi r} \int_0^{2\pi} \int_b^r q r dr d\theta = -\frac{1}{r} \int_b^r q r dr$$

(4) Governing equation

$$\nabla^4 w = \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\right)w = \frac{q}{D}$$

(b is the radius of a central hole, for solid plate, $b = 0$)

This equation is very useful
• Boundary conditions - to determine the constants of integral of the governing equation

(1) Along a clamped edge,

\[
w = \frac{dw}{dr} = 0
\]

(2) Along a simply supported edge,

\[w = M_r = 0\]

(3) Along a free edge,

\[Q_r = M_r = 0\]

(4) At \(r = a\),

\[M_r = M_o\]
Example solutions

(1) Plate with Edge Moment

From the relation:
\[
\frac{d}{dr} \left( \frac{1}{r} d \left( r \frac{dw}{dr} \right) \right) = -\frac{Q_r}{D}
\]

since \(Q_r = 0\) in this case (\(q = 0\)), the above equation yields
\[
\frac{d}{dr} \left( r \frac{dw}{dr} \right) = C_1 r
\]

the second and third integration yield, respectively,
\[
\frac{dw}{dr} = \frac{C_1 r}{2} + \frac{C_2}{r}, \quad w = \frac{C_1 r^2}{4} + C_2 \ln r + C_3
\]

at \(r = 0, \frac{dw}{dr} \neq \infty, \rightarrow C_2 = 0\) (must be)

If the center does not displace, then \(w = 0\) at \(r = 0 \rightarrow C_3 = 0\)

At \(r = a, M_r = M_0\), substituting \(w = \frac{C_1 r^2}{4}\) into
\[
M_r = -D \left( \frac{d^2 w}{dr^2} + \frac{v}{r} \frac{dw}{dr} \right)
\]

we find
\[
\begin{cases}
M_0 = -D \left( \frac{C_1}{2} + \frac{C_1}{2} \right) \rightarrow C_1 = -\frac{2M_0}{(1+v)D} \\
w = -\frac{M_0 r^2}{2(1+v)D}
\end{cases}
\]

(This is bending to a spherical surface, we can check that \(M_r = M_0 = M_0\) throughout the plate. Note that constant \(C_3\) represents only the rigid body translation along \(z\) axis)
(2) Plate with Concentrated Center Force and Clamped Edge

In this case, \[ Q_r = -\frac{P}{2\pi r} \]

substituting into equation
\[ \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) \right] = -\frac{Q_r}{D} \]

and after three integration, we have

\[ w = \frac{P}{8\pi r} \left( r^2 \ln r - r^2 \right) + \frac{C_1 r^2}{4} + C_2 \ln r + C_3 \]

At \( r = 0 \), \( w \neq \infty \), so we must have \( C_2 = 0 \)

At \( r = a \), \( \frac{dw}{dr} = 0 \) \( \rightarrow C_1 = \frac{P}{4\pi D} (1 - 2 \ln a) \)

At \( r = a \), \( w = 0 \) \( \rightarrow C_3 = \frac{Pa^2}{16\pi D} \)

Finally we have

\[ w = \frac{P}{16\pi D} \left( 2r^2 \ln \frac{r}{a} + a^2 - r^2 \right) \]

\[ M_r = -\frac{P}{4\pi} \left[ 1 + (1+v) \ln \frac{r}{a} \right] \]

\[ M_\theta = -\frac{P}{4\pi} \left[ v + (1+v) \ln \frac{r}{a} \right] \]

Discussion:

(1) \( w_{\text{max}} = \frac{Pa^2}{16\pi D} \), at \( r = 0 \)

(2) \( Q_r, M_r, M_\theta \) and \( \frac{dw}{dr} \) all \( \rightarrow \infty \) when \( r \rightarrow \infty \)

This is due to the “concentrated” force which does not really exist, in reality, \( P \) is distributed as \( qdA \), then there is no such “\( \infty \)” phenomena.
(3) Other loadings and boundary conditions

For inner part, \( Q_r = 0 \)

For outer part, \( Q_r = -\frac{q_o}{2r} (r^2 - c^2) \)

By using equation

\[
\frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) \right] = -\frac{Q_r}{D}
\]

we get “inner” and “outer” solutions, there are a total of 6 constants of integration, solved by 6 boundary conditions:

For inner solution: \( M_r = 0 \) at \( r = b \)

For outer solution: \( M_r = 0, \ w = 0 \), at \( r = a \)

at \( r = c \), \( w_{(inner)} = w_{(outer)} \)

\[
\frac{dw}{dr}_{(inner)} = \frac{dw}{dr}_{(outer)}
\]

\[
M_r_{(inner)} = M_r_{(outer)}
\]
(4) Finite element solution

Usually, no analytical solution is available if the loading lacks axial symmetry - then the problem can be solved numerically by **finite element method**.

**FIGURE 4.3.3.** Circular plate, clamped over half its boundary and loaded by a 100-N transverse force at one edge. Diameter = 100mm, thickness = 2.0mm, $E = 200$GPa, $\nu = 0.27$. Contours of stresses $\sigma_x$, $\sigma_y$, $\sigma_{\text{max}}$ on the upper surface, and of lateral deflection $w$, are numerically equidistant.
• Solution by superposition

Example: Determine the deflection \( w \) of a simply supported circular plate loaded by a concentrated center force \( P \).

(a) Solution\(^{(a)}\) = Solution\(^{(b)}\) + Solution\(^{(c)}\).

(b) Clamped edge, at \( r = a \),

\[
M = -\frac{P}{4\pi}, \quad M_\theta = ..., \quad w = ...
\]

(c) Plate with edge moment \( M_o = +\frac{P}{4\pi}, \) using the previous solution (note here the plate is simply supported, the original solution for \( w \) should be translated downward an amount equal to \( w \) at \( r = a \)), we have

\[
w = -\frac{M_o r^2}{2(1+\nu)D} + \frac{M_o a^2}{2(1+\nu)D}, \quad \text{amount of translation, so that } w = 0 \text{ at } r = a
\]

\[
M_\theta = M_o = \frac{P}{4\pi}
\]

Superposition results of (b) and (c), we obtain result for (a):

\[
w = -\frac{P}{16\pi D} \left[ 3 + \nu \left( a^2 - r^2 \right) + 2r^2 \ln \frac{r}{a} \right]
\]

\[
M_r = -\frac{P}{4\pi} (1+\nu) \ln \frac{r}{a}, \quad M_\theta = -\frac{P}{4\pi} \left[ -1 + \nu + (1+\nu) \ln \frac{r}{a} \right]
\]
A short catalog of solutions:


Expressions for $w$, $M_r$, and $M_\theta$ as functions of $r^*$ are given by Eqs. 4.3.6, 4.3.7, and 4.3.8.

At $r = 0$: $\quad w = \frac{Pa^2}{16\pi D} \quad M_r = M_\theta = \infty$ \hspace{1cm} (see footnote 3)

At $r = a$: $\quad M_r = -0.0796P \quad M_\theta = -0.0239P$

Case 2. Circular plate with simply supported edge, concentrated center force $P$.

Expressions for $w$, $M_r$, and $M_\theta$ as functions of $r$ are given by Eqs. 4.4.3, 4.4.4, and 4.4.5.

At $r = 0$: $\quad w = 0.0505 \frac{Pa^2}{D} \quad M_r = M_\theta = \infty$ \hspace{1cm} (see footnote 3)

At $r = a$: $\quad \frac{dw}{dr} = -0.0612 \frac{Pa}{D} \quad M_\theta = 0.0557P$
Case 3. Circular plate with clamped edge, uniform pressure $q_o$.

\[
w = \frac{q_o}{64D} (a^2 - r^2)^2
\]

\[
M_r = \frac{q_o}{16} [(1 + \nu)a^2 - (3 + \nu)r^2]
\]

At $r = 0$:
\[
w = \frac{q_o a^4}{64D}
\]
\[
M_r = M_\theta = 0.0813 q_o a^2
\]

At $r = a$:
\[
M_r = -0.125 q_o a^2
\]
\[
M_\theta = -0.0375 q_o a^2
\]

\(^3\)Central bending moments in Cases 1 and 2 are infinite according to classical thin-plate theory. Adjustments suggested in footnote 1 yield more realistic bending moments at $r = r_o = 0$, namely

\[
M_r = M_\theta = \frac{P}{4\pi} \left[ (1 + \nu) \ln \frac{a}{0.325t} \right]
\]

if clamped (Case 1)

\[
M_r = M_\theta = \frac{P}{4\pi} \left[ 1 + (1 + \nu) \ln \frac{a}{0.325t} \right]
\]

if simply supported (Case 2)
Case 4. Circular plate with simply supported edge, uniform pressure $q_o$.

\[ w = \frac{q_o(a^2 - r^2)}{64D} \left( \frac{5 + \nu}{1 + \nu} a^2 - r^2 \right) \]

\[ M_r = \frac{q_o}{16} (3 + \nu)(a^2 - r^2) \]

At $r = 0$: \[ w = 0.0637 \frac{q_o a^4}{D} \quad M_r = M_{\theta} = 0.2063 q_o a^2 \]

At $r = a$: \[ \frac{dw}{dr} = -0.0962 \frac{q_o a^3}{D} \]

Case 5. Circular plate, outer edge simply supported, moments $M_r = M_1$ and $M_r = M_2$ distributed around the inner and outer edges, respectively.

\[ w = C_1 \frac{r^2}{4} + C_2 \ln \frac{r}{a} + C_3 \]

\[ \frac{dw}{dr} = C_1 \frac{r}{2} + C_2 \frac{1}{r} \]

where

\[ C_1 = -\frac{2(a^2M_2 - b^2M_1)}{(1 + \nu)(a^2 - b^2)D} \quad C_2 = -\frac{a^2b^2(M_2 - M_1)}{(1 - \nu)(a^2 - b^2)D} \]

\[ C_3 = -\frac{C_1 a^2}{4} \]

If $b = 0$, then $M_r = M_{\theta} = M_2$ throughout the plate, and

\[ w = 0.3846 \frac{M_2 a^2}{D} \quad \text{at } r = 0 \quad \frac{dw}{dr} = -0.7692 \frac{M_2 a}{D} \quad \text{at } r = a \]
Case 6. Circular plate, outer edge simply supported, line load $V$ (force per unit length) around the edge of a central hole of radius $b$.

\[
\begin{align*}
  w &= g_1 \frac{Va^3}{D} \\
  \frac{dw}{dr} &= g_2 \frac{Va^2}{D} \\
  M_{\max} &= M_{\theta b} = g_3 Va
\end{align*}
\]

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<th>0.5</th>
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Case 7. Circular plate, outer edge simply supported, inner edge free, uniform pressure $q_o$.

\[
\begin{align*}
  w &= g_4 \frac{q_o a^4}{D} \\
  \frac{dw}{dr} &= g_5 \frac{q_o a^3}{D} \\
  M_{\max} &= M_{\theta b} = g_6 q_o a^2
\end{align*}
\]

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Further examples of Superposition

FIGURE 4.6.1. Examples of superposition solutions. (a) Circular plate with inner edge simply supported, outer edge free, uniform pressure $q_o$. (b) Circular plate with outer edge clamped, line load $V$ (force per unit length) around the edge of a central hole of radius $b = a/2$. (c) Solid circular plate, outer edge simply supported, line load $V$ (force per unit length) at radius $r = 0.3a$. 
Find the maximum deflection of the plate \((r = a)\)

\[
\begin{align*}
&= & & +
\end{align*}
\]

case 6  \quad \text{(Eq. 4.6.1)}

Find the maximum flexural stress in the plate \((r = a)\)

\[
\begin{align*}
&= & & +
\end{align*}
\]

case 6  \quad \text{(Eq. 4.6.3)}

Find the center deflection in the plate \((r = 0)\)

\[
\begin{align*}
&= & & +
\end{align*}
\]

case 5 & 6