CHAPTER FOUR
ELASTIC FOUNDATIONS

*** Bending of beams on elastic foundations and solutions**

**** Solution by superposition and Contact stress problems**

4.1 Introduction and Foundation Models ---- Winkler Foundation

4.2 Governing Equations For Uniform Straight Beams on Elastic Foundations

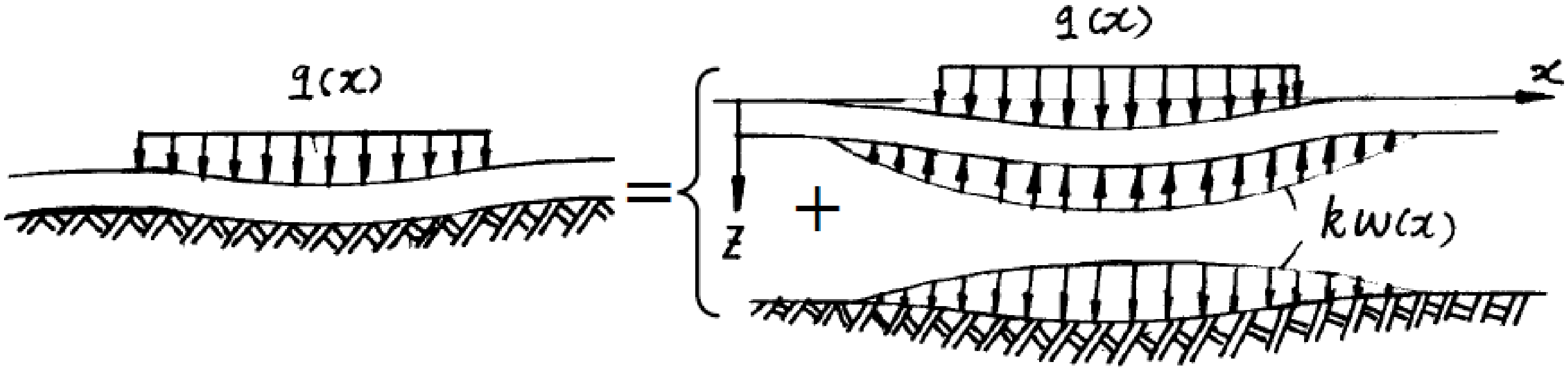
4.3 Semi-infinite and Infinite Beams with Concentrated Loads

4.4 Semi-infinite and Infinite Beams with Distributed Loads, Short Beams

4.5 Contact Stresses ---- Problem and Solutions

4.1 Introduction and Foundation Models - Winkler Foundation

- Concept of Elastic Foundations and the Effect of the Foundation on the Beam (a kind of contact)



* Not to study the stresses in the foundation itself.

- Two Analytical **Models on Elastic Foundation**

(1) Model 1 - **Winkler Model** - a linear force-deflection relationship is presumed

FIGURE 5.1.1. Deflections of foundation models under uniform pressure. No beam is present.

- Two Analytical **Models on Elastic Foundation**

(1) Model 1 - **Winkler Model** - a linear force-deflection relationship is presumed

FIGURE 5.1.1. Deflections of foundation models under uniform pressure. No beam is present.

A linear relationship between the force on the foundation (pressure p) and the deflection w is assumed:

$\mathbf{p = k_o w}$ k_o is the foundation modulus (unit: $N/m^2/m$)

For beams with width b, we use
$\mathbf{p = kw = k_o bw}$**, unit of k: N/m/m**

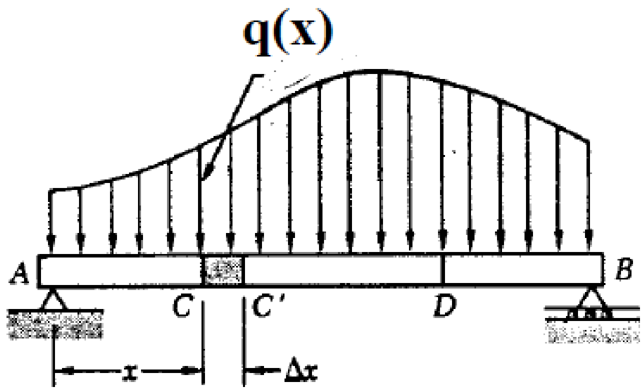
** An Important restriction of the model: the contact is never broken between beam and foundation

(2) Model 2 ---- Elastic solid Foundation ---- More realistic but bore complicated (not used here)

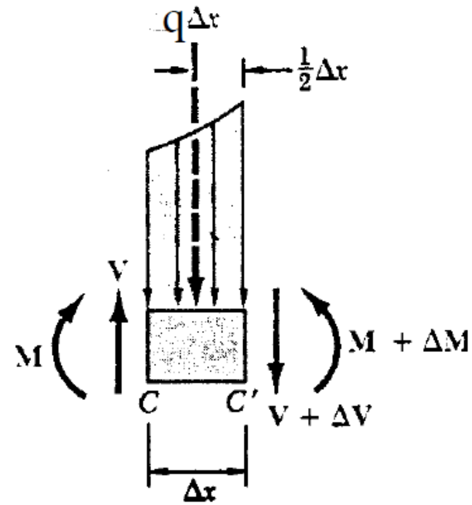
4.2 Governing Equations For Uniform Straight Beams on Elastic Foundations

- Governing Equations

(1) In **Usual Beam Theory (MECH 101)**



(a)



(b)

$$\frac{dV}{dx} = -q, \frac{dM}{dx} = V \rightarrow \frac{d^2M}{dx^2} = -q$$

$$M = -EI\frac{d^2w}{dx^2} \rightarrow EI\frac{d^4w}{dx^4} = q$$

(2) Beam Theory on Winkler Foundation

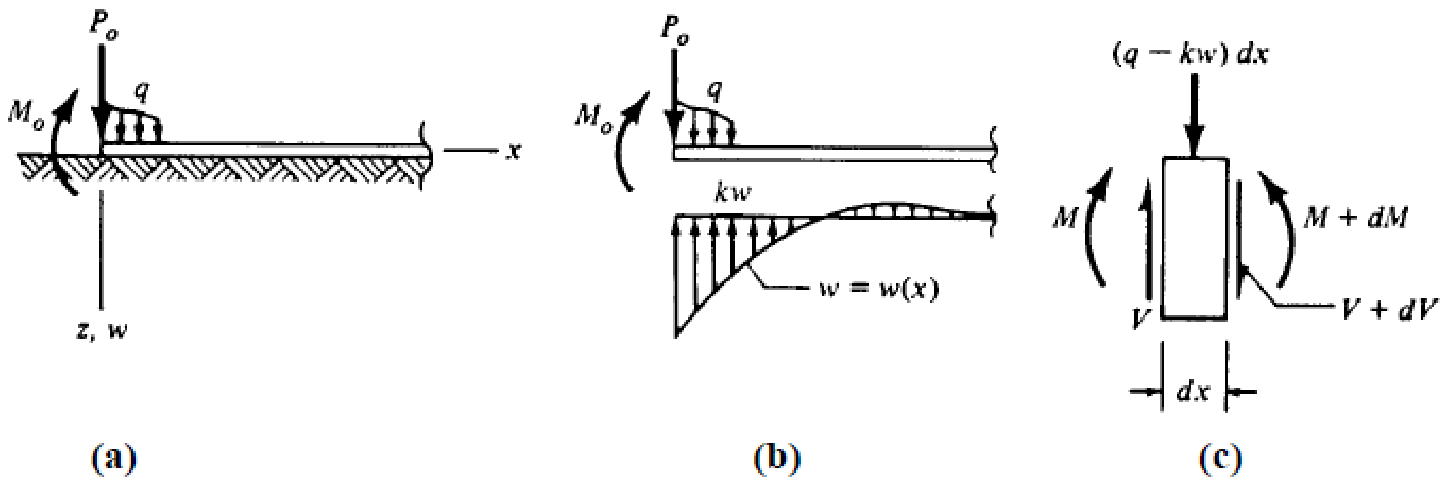


FIGURE 5.2.1. (a) Arbitrary loading on an elastically supported beam. (b) Reaction kw of a Winkler foundation. The curve w = w(x) is the deflected shape of the beam. (c) Forces that act on a differential element of the beam.

$$\frac{dV}{dx} = -q + kw, \frac{dM}{dx} = V \rightarrow \frac{d^2M}{dx^2} = kw - q$$

$$M = -EI\frac{d^2w}{dx^2} \rightarrow EI\frac{d^4w}{dx^4} + kw = q$$

- **Solution of the Equation**

The governing equation for a uniform beam on Winkler foundation:

$$EI\frac{d^4w}{dx^4} + kw = q$$

By introducing a parameter b (unit L^{-1})

$$\beta = \left[\frac{k}{4EI}\right]^{\frac{1}{4}}$$

The solution of the governing equation can be written as

$$w = e^{\beta x}(C_1 \sin \beta x + C_2 \cos \beta x) + e^{-\beta x}(C_3 \sin \beta x + C_4 \cos \beta x) + \underline{w(q)}$$

Particular solution related with q, w(q) = 0 when q = 0

C_1, C_2, C_3, C_4 are constants of integration, which are determined by B.C. When w(x) is known, V, M, θ, σ, etc can be calculated by the relevant formulas.

For the convenience, the following symbols are defined:

$$A_{\beta x} = e^{-\beta x}(\cos \beta x + \sin \beta x), B_{\beta x} = e^{-\beta x} \sin \beta x$$

$$C_{\beta x} = e^{-\beta x}(\cos \beta x - \sin \beta x), D_{\beta x} = e^{-\beta x} \cos \beta x$$

These quantities are related by certain derivatives, and the value of the above quantities are listed in the table.

TABLE 5.2.1 Selected Values of Terms Defined by Eqs. 5.2.7.

βx	$A_{\beta x}$	$B_{\beta x}$	$C_{\beta x}$	$D_{\beta x}$
0	1	0	1	1
0.02	0.9996	0.0196	0.9604	0.9800
0.04	0.9984	0.0384	0.9216	0.9600
0.10	0.9907	0.0903	0.8100	0.9003
0.20	0.9651	0.1627	0.6398	0.8024
0.30	0.9267	0.2189	0.4888	0.7077
0.40	0.8784	0.2610	0.3564	0.6174
0.50	0.8231	0.2908	0.2415	0.5323
0.60	0.7628	0.3099	0.1431	0.4530
0.70	0.6997	0.3199	0.0599	0.3798
$\pi/4$	0.6448	0.3224	0	0.3224
0.80	0.6354	0.3223	−0.0093	0.3131
0.90	0.5712	0.3185	−0.0657	0.2527
1.00	0.5083	0.3096	−0.1108	0.1988
1.10	0.4476	0.2967	−0.1457	0.1510
1.20	0.3899	0.2807	−0.1716	0.1091
1.30	0.3355	0.2626	−0.1897	0.0729
1.40	0.2849	0.2430	−0.2011	0.0419
1.50	0.2384	0.2226	−0.2068	0.0158
$\pi/2$	0.2079	0.2079	−0.2079	0
1.60	0.1959	0.2018	−0.2077	−0.0059
1.70	0.1576	0.1812	−0.2047	−0.0235
1.80	0.1234	0.1610	−0.1985	−0.0376
1.90	0.0932	0.1415	−0.1899	−0.0484
2.00	0.0667	0.1231	−0.1794	−0.0563
2.20	0.0244	0.0896	−0.1548	−0.0652
$3\pi/4$	0	0.0670	−0.1340	−0.0670
2.40	−0.0056	0.0613	−0.1282	−0.0669
2.60	−0.0254	0.0383	−0.1019	−0.0636
2.80	−0.0369	0.0204	−0.0777	−0.0573
3.00	−0.0423	0.0070	−0.0563	−0.0493
π	−0.0432	0	−0.0432	−0.0432
3.20	−0.0431	−0.0024	−0.0383	−0.0407
3.40	−0.0408	−0.0085	−0.0237	−0.0323
3.60	−0.0366	−0.0121	−0.0124	−0.0245
3.80	−0.0314	−0.0137	−0.0040	−0.0177
$5\pi/4$	−0.0279	−0.0139	0	−0.0139
4.00	−0.0258	−0.0139	0.0019	−0.0120
$3\pi/2$	−0.0090	−0.0090	0.0090	0
2π	0.0019	0	0.0019	0.0019

4.3 Semi-infinite and Infinite Beams with Concentrated Loads

- **Semi-infinite beams** with **concentrated load**

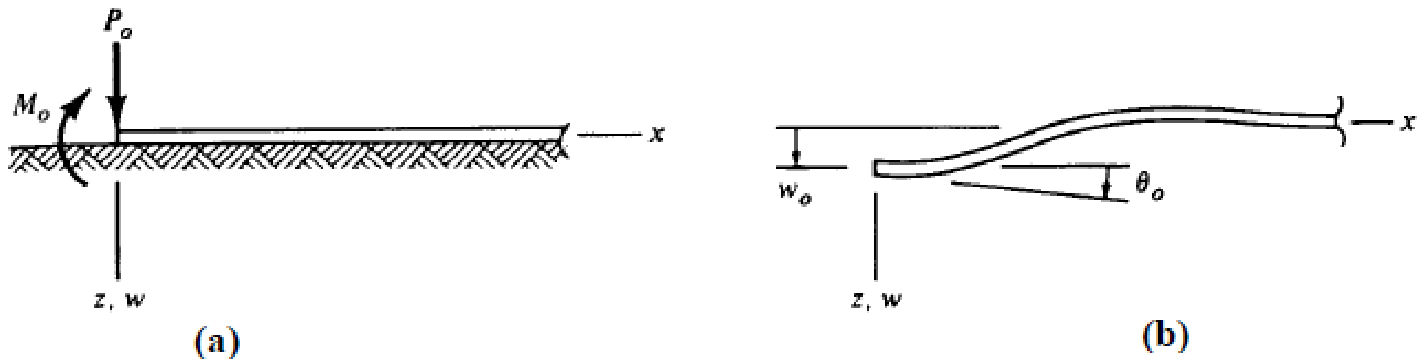


FIGURE 5.3.1. (a) Concentrated loads P_o and M_o at the end of a semi-infinite beam on a Winkler foundations. (b) End deflection w_o and end rotation θo = $(dw/dx)_{x=0}$, both shown in the positive sense.

- **Two kind of boundary conditions:**

(1) prescribe P_o and M_o at x = 0

(2) prescribe w_o and θ_o at x = 0

- **For Boundary condition (1)**

Let w(q) = 0 in the general expression of solution. Since w = 0 at x →∞ , we must have $C_1 = C_2 = 0$. The other two boundary conditions determine C_3, C_4.

$$M\Big|_{x=0} = -EI\frac{d^2w}{dx^2}\Big|_{x=0} = M_o \rightarrow C_3 = \frac{2\beta^2 M_o}{k}$$

$$V\Big|_{x=0} = -EI\frac{d^3w}{dx^3}\Big|_{x=0} = 0 = -P_o \rightarrow C_4 = \frac{2\beta P_o}{k} - \frac{2\beta^2 M_o}{k}$$

So finally we have:

$$w(x) = \frac{2\beta P_o}{k} D_{\beta x} - \frac{2\beta M_o}{k} C_{\beta x},$$

$$\theta = \frac{dw}{dx} = -\frac{2\beta^2 P_o}{k} A_{\beta x} + \frac{4\beta^3 M_o}{k} D_{\beta x}, M(x), V(x)$$

All M(x), V(x), w(x), q(x) are damped sine and cosine wave

• **Example**

A semi-infinite steel bar (E = 200GPa) has a square cross section (b = h = 80mm) and rests on a Winkler foundation of modulus k_o = 0.25 N/mm²/mm. A downward force of 50kN is applied to the end. Find the maximum and minimum deflections and their locations. Also find max. flexural stress and its location.

(1) Necessary constants are:

$$EI = 200000\frac{80^4}{12} = 6.827x10^{11} N\cdot mm^2, k = 80k_0 = \frac{20N}{mm\cdot mm}, \rightarrow \beta = \left[\frac{k}{4EI}\right]^{\frac{1}{4}} = 0.001645/mm$$

(2) The displacement w(x) = $2\beta P_o D_{\beta x}$ / k

$$w_{\max} = w\big|_{x=0} = w_o = \frac{2\beta P_o}{k} = 8.225mm$$

The min. deflection occurs at the smallest distance for which q = 0. From $\theta = -\frac{2\beta^2 P_o}{k} A_{\beta x}$ we find $A_{\beta x}$ = 0 at βx = 3p/4 or x = 1432mm, corresponding $D_{\beta x}$ = -0.0670, so

$$w_{\min} = \frac{2\beta P_o D_{\beta x}}{k} = -0.551mm$$

(This upward deflection reminds us our assumption on the beam – foundation connection)

(3) Bending moment is M = $-P_o B_{\beta x}$ / b, from the table we find that Bbx has largest value at βx = π /4, the corresponding $B_{\beta x}$ = 0.3234, so Mmin = -9.8 x 10^6 Nmm

$$\rightarrow \quad \sigma_{\max} = \frac{Mc}{I} = 115MPa$$ appears on top of the beam at x = π /4β = 477mm.

- **Infinite beams** with **concentrated load**

(1) Concentrated force - by using previous solution - equivalent to:

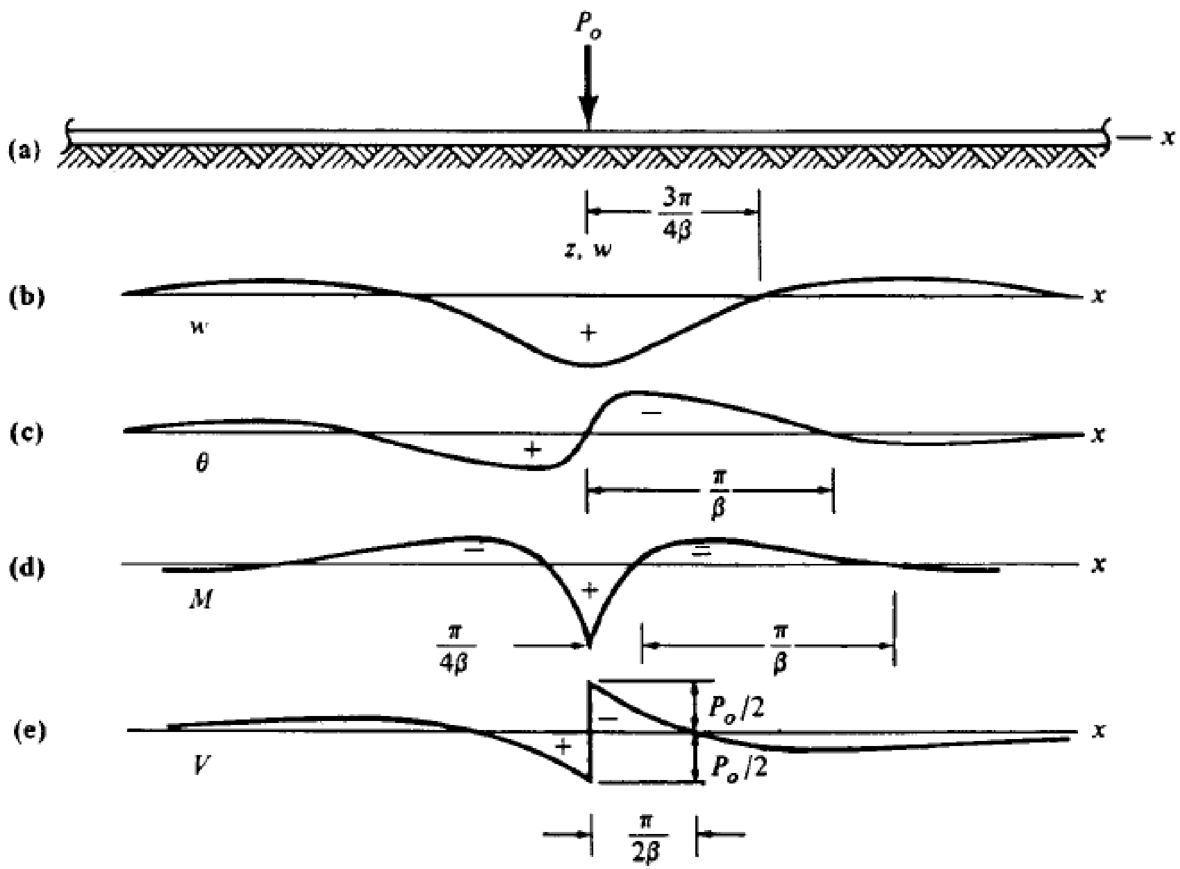


FIGURE 5.4.1. (a) Concentrated load P_o at x = 0 on a uniform infinite beam that rests on a Winkler foundation. (b-e) Curves for deflections, rotation, bending moment, and transverse shear force in the beam. These curves are proportional to $A_{\beta x}$, $B_{\beta x}$, $C_{\beta x}$, $D_{\beta x}$, respectively.

By using the solution for semi-infinite beam under concentrated load, we have:

at x = 0, $\theta = -\frac{2\beta^2\left(P_o/2\right)}{k} + \frac{4\beta^2 M_o}{k} = 0 \rightarrow M_o = \frac{P_o}{4\beta}$ due to symmetry

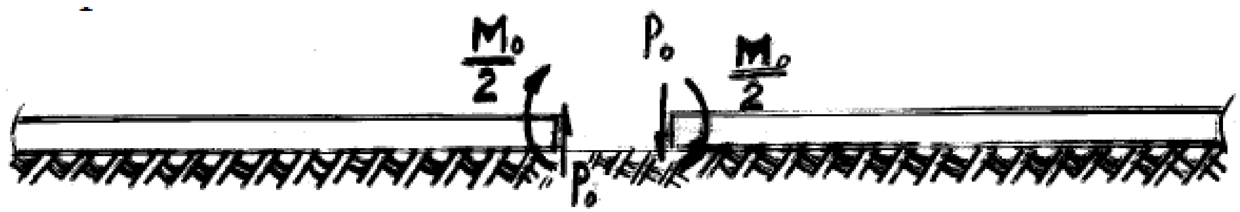
(mirror at x = 0), we have V = 0 at x = 0. Substituting P_o / 2 and M_o = P_o / 4β in the previous solution (semi-infinite beam under concentrated force and moment at the end), we obtain the solution for infinite beam here:

$$w = \frac{\beta P_o}{2k} A_{\beta x}; \theta = \frac{dw}{dx} = -\frac{\beta^2 P_o}{k} B_{\beta x};$$

$$M = \frac{P_o}{4\beta} C_{\beta x}; V = -\frac{P_o}{2} D_{\beta x}$$

Notes: In these solutions, x should be x ≧ 0, for x < 0, the w(x), M(x), θ(x) and V(x) must be obtained from the symmetry and antisymmetry conditions: w(x) = w(-x), θ(x) = -θ(-x), M(x) = M(-x), V(x) = -V(-x).

(2) Concentrated moment - by using previous solution - equivalent to:



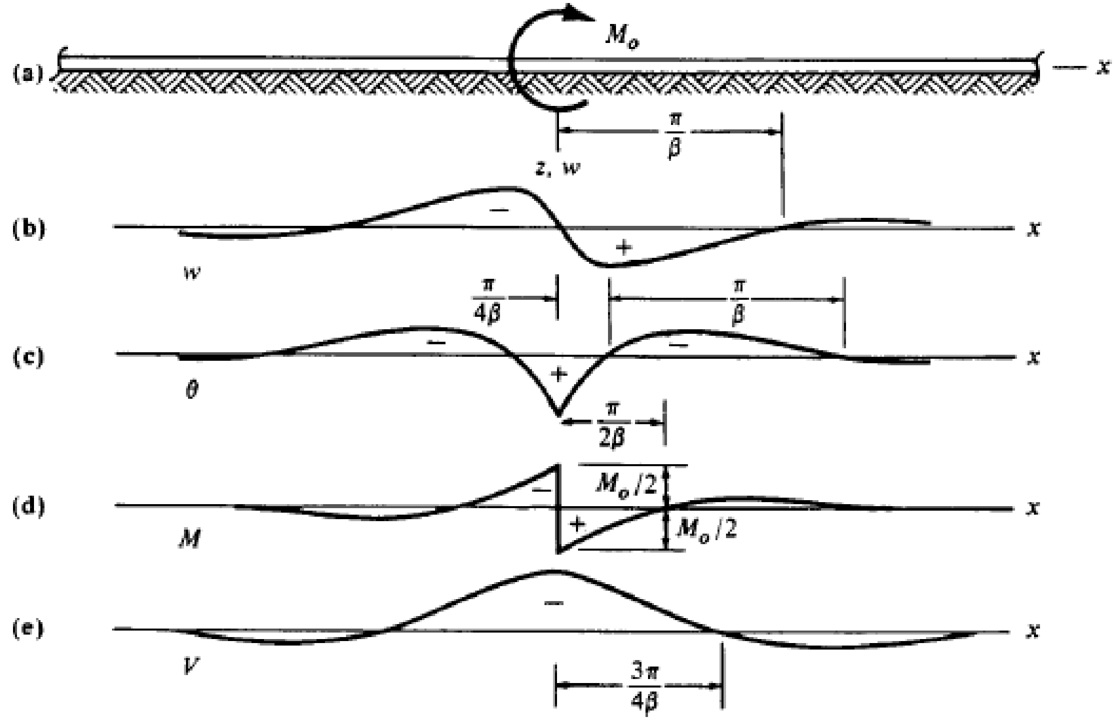


FIGURE 5.4.2. (a) Concentrated moment M_o at x = 0 on a uniform infinite beam that rests on a Winkler foundation. (b-e) Curves for deflection, rotation, bending, moment, and transverse shear force in the beam. These curves are proportional to $B_{\beta x}$, $C_{\beta x}$, $D_{\beta x}$, and $B_{\beta x}$', respectively.

(1) Deformation analysis: Deflections are antisymmetric with respect to the origin; so $w|_{x=0}$ = 0. Bending moment $M\big|_{x=0^+} = \frac{M_o}{2}, M\big|_{x=0^-} = -\frac{M_o}{2}$,

Substituting into the expression w(x) for semi-infinite beam with concentrated load,

$$w\big|_{x=0} = 0 = \frac{2\beta P_o}{k} - \frac{2\beta(M_o/2)}{k} \rightarrow P_o = \frac{M_o\beta}{2}$$

(2) Then substituting $P_o = \frac{M_o\beta}{2}, M = \frac{M_o}{2}$ into basic solution, we have

$$w(x) = \frac{\beta^2 M_o}{k} B_{\beta x}, \theta = \frac{dw}{dx} = \frac{\beta^3 M_o}{k} C_{\beta x}$$

$$M(x) = \frac{M_o}{2} D_{\beta x}, V = -\frac{\beta M_o}{2} A_{\beta x}$$

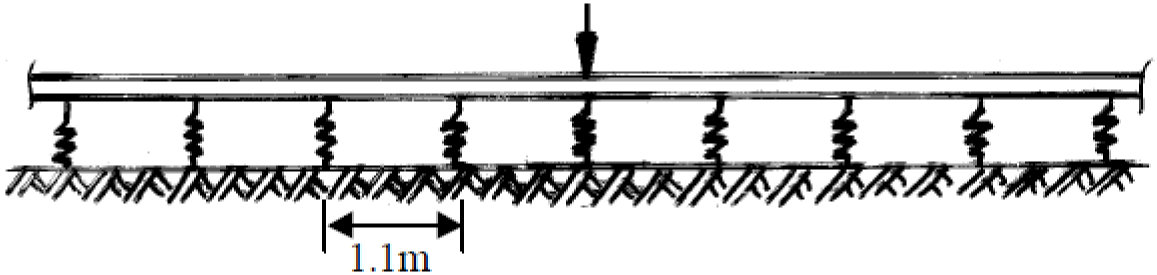
The solutions for the left half of the beam must be obtained from the following symmetry and antisymmetry conditions:

w(x) = -w(x); θ(x) = θ(-x); M(x) = -M(-x);

V(x) = V(-x).

- **Example:**

P = 18kN



A infinite beam rest on equally spaced linear coil springs, located every 1.1m along the beam. A concentrated load of 18kN is applied to the beam, over one of the springs. EI of the beam is 441×10^9 Nmm^2, K = 275 N/mm for each spring. Compute the largest spring force and largest bending moment in the beam.

(1) To "smear" the springs into a Winkler foundation:
force applied to the beam by a spring with deflection w is Kw, so if the spring spacing is L, the associated force in each span L is Kw, then the hypothetical distributed force is therefore Kw / L

distributed force kw of Winkler foundation	=	**distributed force Kw / L by a series of springs**

The "equivalent" Winkler foundation modulus is k = K/L and the $\beta = [k/4EI]^{1/4} = 6.136 \times 10^{-4}$ /mm

(2) According to the previous solution for infinite beam with concentrated load P, we have

$$w_{\max} = w\Big|_{x=0} = \frac{\beta P_o}{2k} A_{\beta x}$$ = 22.1mm, the maximum spring force is Fmax = Kwmax = 6075N

$$M_{\max} = M\Big|_{x=0} = \frac{P_o}{4\beta} C_{\beta x} = 7.33\, kN \cdot m$$

(3) If the beam length is finite with several springs, then the problem can be solved as static indeterminate beam.

- **Example:**

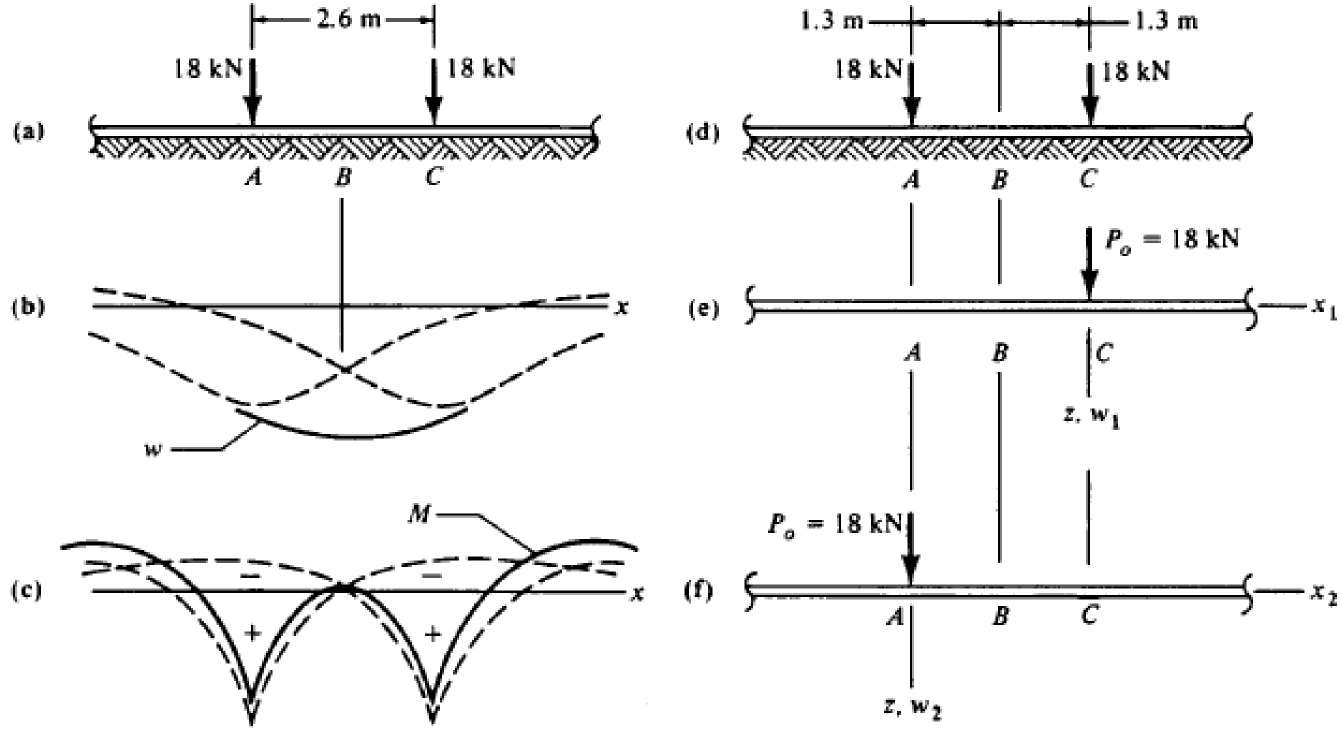


FIGURE 5.4.3. (a) Equal concentrated loads on an elastically supported beam. (b-c) Resulting deflection and bending moment. Dashed lines represent results of individual loads. Solid lines are superposed results. (d-f) Coordinate systems used to solve the problem by superposition.

An infinite beam on a Winkler foundation has the following properties:

$$EI = 441x10^9 N \cdot mm^2, k = 0.25 N/mm/mm, \ \beta = 6.136x10^{-4}/mm$$

Two concentrated loads, 18kN each and 2.6m apart, are applied to the beam. Determine w_{max} and M_{max}. Principal of superposition: total w and M are

$$w = w_1 + w_2; M = M_1 + M_2$$

We find that w_{max} is at point B, M_{max} is at A and C. The resultant w is larger than a single load, but resultant M is a little smaller than the case of a single load.

4.4 Semi-infinite and Infinite Beams with Distributed Loads, Short Beams

- **Semi-infinite beam** with **distributed load** over the **entire span**

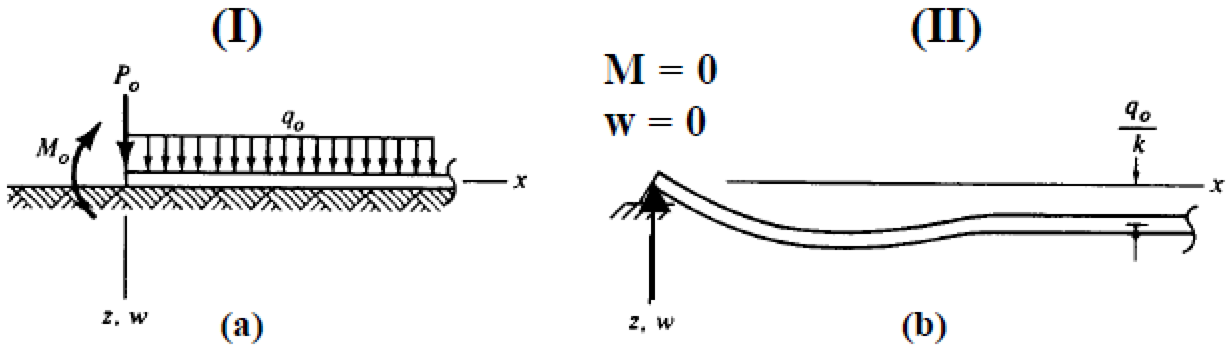


FIGURE 5.5.1. (a) Semi-infinite beam on a Winkler foundation, loaded by end force P_o, end moment M_o, and a uniformly distributed load q_o over the entire beam.
(b) Deflected shape of the beam if simply supported and loaded by q_o only.

(I) Analysis: since q_o is added to the entire beam, we begin with the general solution. At large x, the beam does not bend. There the load is carried by the foundation uniformly with deflection q_o / k. So in the general solution, we have C1 = C2 = 0 and w(q) = q_o / k, and

$$w = C_3 B_{\beta x} + C_4 D_{\beta x} + \underline{\frac{q_o}{k}} \quad \text{(due to } q_o\text{)}$$

The boundary condition at x = 0 leads to

$$M\Big|_{x=0} = M_o \rightarrow C_3 = \frac{2\beta^2 M_o}{k}, V\Big|_{x=0} = -P_o \rightarrow C_4 = \frac{2\beta P_o}{k} - \frac{2\beta^2 M_o}{k}$$

The solutions are finally,

$$w = \frac{2\beta P_o}{k} D_{\beta x} - \frac{2\beta^2 M_o}{k} C_{\beta x} + \underline{\frac{q_o}{k}}$$

(due to q_o)

$$\theta = -\frac{2\beta^2 P_o}{k} A_{\beta x} + \frac{4\beta^3 M_o}{k} D_{\beta x}, M = \ldots, V = \ldots$$

(II) In this case, the boundary conditions are $M|_{x=0} = 0$, $w|_{x=0} = 0$ → C3, C4 → w → support reaction at x = 0 to be $Q_0/2\beta$.

- **Infinite beam with distributed load over a length L**

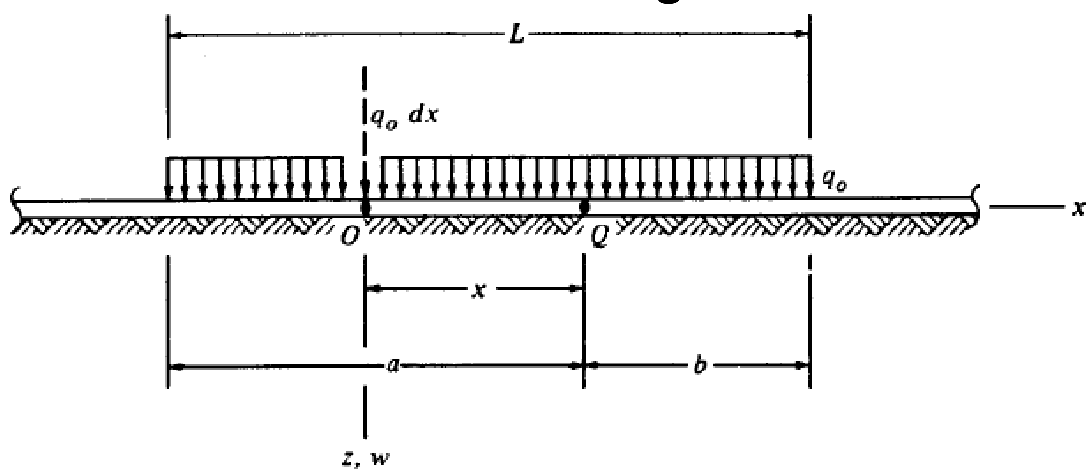


FIGURE 5.5.2. Uniformly distributed load q_o, over a length L = a + b of an infinite beam on a Winkler foundation.

(1) Method: Principal of superposition

(2) Basic solution: infinite beam under concentrated force P

$$w = \frac{\beta P_o}{2k} A_{\beta x}$$

(3) The deflection at Q due to load qodx at O is

$$dw_Q = \frac{\beta q_o dx}{2k} A_{\beta x}$$

(4) The total deflection at Q is

$$w_Q = \frac{\beta q_o}{2k}\left[\int_0^a A_{\beta x} dx + \int_0^b A_{\beta x} dx\right] = -\frac{q_o}{2k}\left[D_{\beta x}\Big|_0^a + D_{\beta x}\Big|_0^b\right]$$

$$w_Q = \frac{q_o}{2k}\left(2 - D_{\beta a} - D_{\beta b}\right)$$

(5) By the same integration, we get the total M at Q

$$M_q = \frac{q_o}{4\beta^2}\left(B_{\beta a} + B_{\beta b}\right)$$

$$\text{and } \theta_Q = \frac{\beta q_o}{2k}\left(A_{\beta a} - A_{\beta b}\right), V_Q = \frac{q_o}{4\beta}\left(C_{\beta a} - C_{\beta b}\right)$$

It is helpful to identify three cases:

(I) β L is small (or β is small), L is small: The deflection and bending moment are greatest at the middle of the span L, the corresponding condition is that $\beta L < \pi$.
(II) β L is large: (1) deflection is constant in the center portion $w = q_o / k$, and bending moment is zero except in the neighborhood of the ends of the loaded zone.
(III) Intermediate values of βL. $\pi < \beta L$

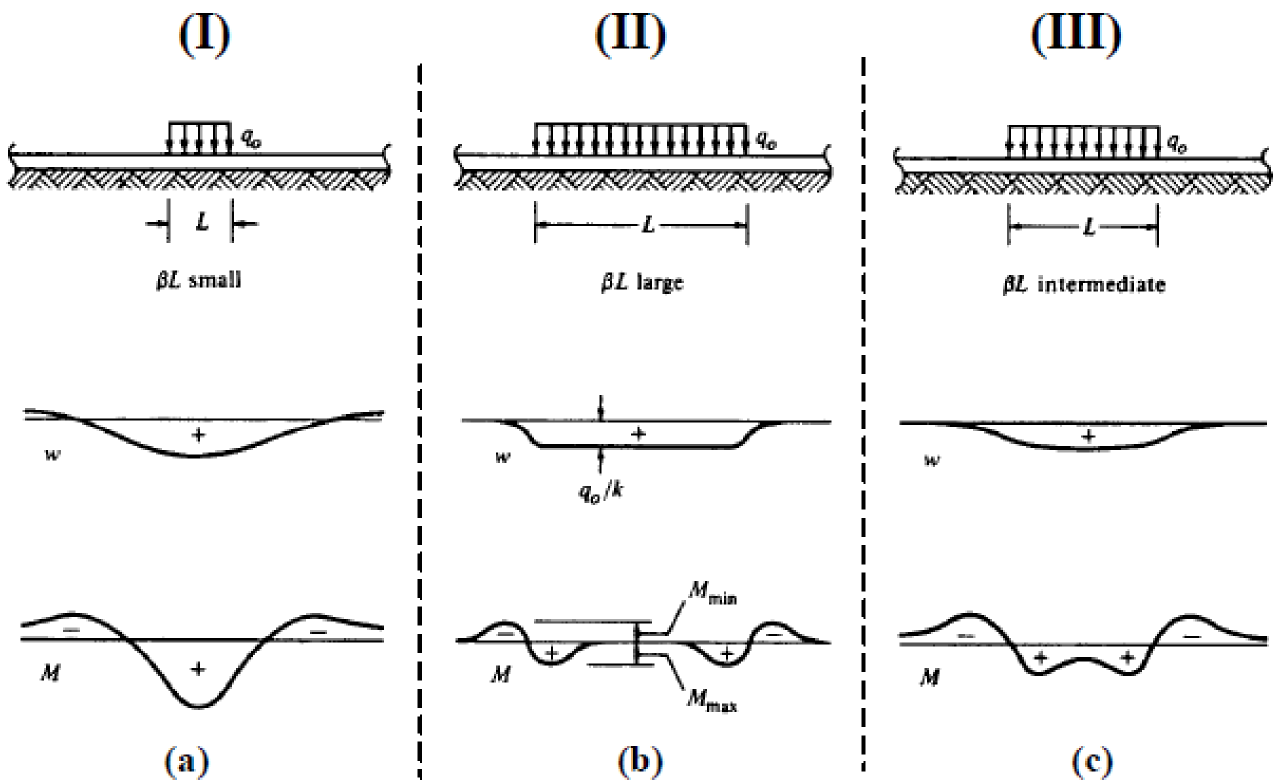


FIGURE 5.5.3 Deflection and bending moment in uniform and uniformly loaded infinite beams on a Winkler foundation.

• Short beams on a Winkler foundation

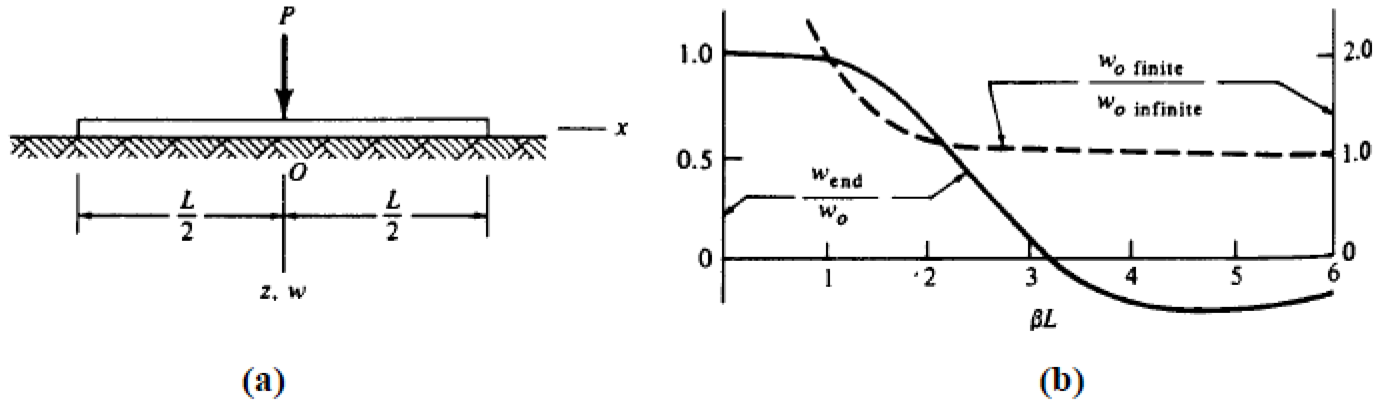


FIGURE 5.6.1. (a) Centrally loaded beam of finite length on a Winkler foundation. (b) End deflection w_{end} at $x = \pm L / 2$, as a fraction of center deflection w_o, versus βL. Also, the ratio of w_o for a finite beam to w_o for an infinitely long beam.

(1) Four boundary conditions: At x = 0, θ = 0 and $V = -\frac{P}{2}$

at $x = \frac{L}{2}, -\frac{L}{2}$, M = V = 0

(2) Get four constants C1, C2, C3, C4, the results are known and are tabulated for several cases.

(3) Also 3 cases can be classified: (a) short beams; (b) intermediate beams; (c) long beams. The ratio of center deflection changes with the length of the beam. The ratio of end deflection to center deflection is also plotted in the figure.

4.5 Contact Stress ---- Problem and Solutions

- Features of the contact problem

(1) The area of contact between bodies grows as load increases
(2) In the contact stress problem, stresses remain finite

- The pioneer work by Hertz in 1881

- Basic assumption:

(1) The contacting bodies are linearly elastic, homogeneous, isotropic, and contacting zone is relatively small.
(2) Friction is taken as zero → contact pressure is normal to the contact area.

- Solution for two contacting spheres

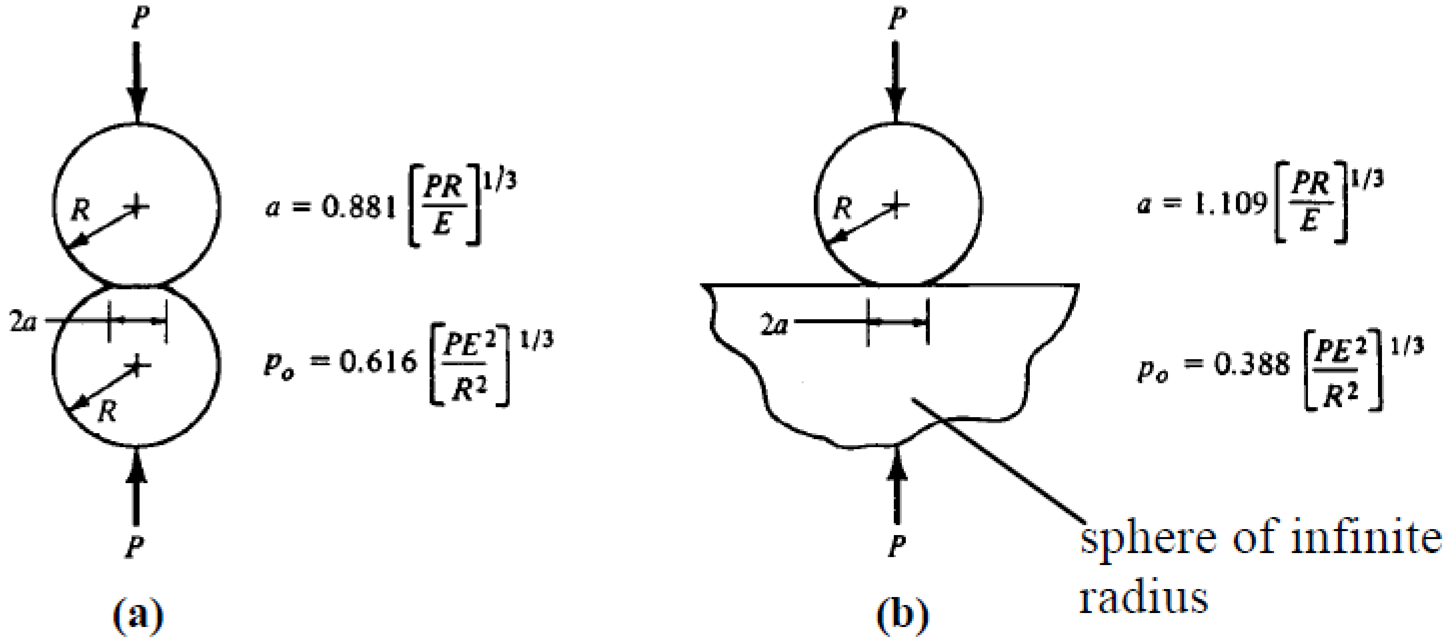


FIGURE 5.8.1. Radius a of the contact area and peak contact pressure po for the cases of (a) two spheres of equal radius, and (b) a sphere on a half-space (which amounts to a sphere of infinite radius). Poisson's ratio is taken as υ = 0.3.

Contact area: circle of radius $a = 1.109\left[\frac{P}{E}\left(\frac{R_1R_2}{R_1+R_2}\right)\right]^{\frac{1}{3}}$

The maximum contact pressure $p_o = \frac{3}{2}\frac{P}{\pi a^2} = 0.388\left[PE^2\left(\frac{R_1+R_2}{R_1R_2}\right)^2\right]^{\frac{1}{3}}$

when a sphere (R_1) pressed into a spherical socket (R_2), $R_2 > R_1$, the results are obtained by making R_2 negative!

Solution for two parallel contact cylinders of length L (L ≧ 10a)

(1) Contact area: long rectangle L x 2a

$$(2)\ a = 1.52\sqrt{\frac{P}{LE}\left(\frac{R_1R_2}{R_1+R_2}\right)},\ p_o = 0.418\sqrt{\frac{PE}{L}\left(\frac{R_1+R_2}{R_1R_2}\right)}$$

- Solution for two crossed cylinders ($R_1 = R_2$)
  (1) Contact area: circular
  (2) a and po are obtained from equations in Fig. 5.8.1(b)
- Some discussions
  (1) Contact pressure is not proportional to P
  (2) Stress state in the center of the contact area between spheres (x = y = z = 0)

$$\sigma_x = \sigma_y = -\frac{1+2\upsilon}{2}p_o, \sigma_z = -p_o$$

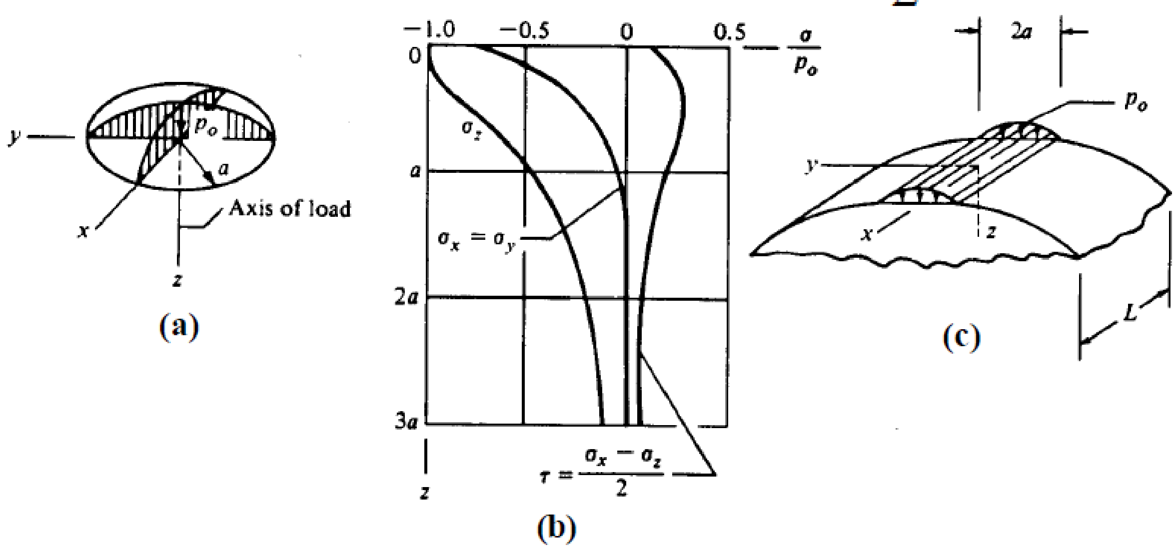


FIGURE 5.8.2. (a) Circular contact area between two spheres. Contact pressure varies quadratically from a maximum of p_o at x = y = 0.
(b) Principal stresses and maximum shear stress along the axis of loads P in contacting spheres, for υ = 0.3.
(c) Rectangular contact area between parallel cylinders.