Chapter 6  Plane Problems on Thermal Stresses

6.1 Thermoelastic Physical Equations

We denote \( T (x, y, z) \) the temperature change, i.e., the final temperature minus the initial temperature, \( \alpha \) the coefficient of thermal expansion of a thermally isotropic material, the thermal strain components are

\[
\varepsilon_x = \varepsilon_y = \varepsilon_z = \alpha T, \quad \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0
\]

The thermal stress (3-D case). The thermally induced strains can be superimposed to those induced by stresses and the total strain will be:

\[
\varepsilon_x = \frac{1}{E} \left[ \sigma_x - \mu (\sigma_y + \sigma_z) \right] + \alpha T \\
\varepsilon_y = \frac{1}{E} \left[ \sigma_y - \mu (\sigma_z + \sigma_x) \right] + \alpha T \\
\varepsilon_z = \frac{1}{E} \left[ \sigma_z - \mu (\sigma_x + \sigma_y) \right] + \alpha T \\
\gamma_{yz} = \frac{2(1 + \mu)}{E} \tau_{yz}, \quad \gamma_{zx} = \frac{2(1 + \mu)}{E} \tau_{zx}, \quad \gamma_{xy} = \frac{2(1 + \mu)}{E} \tau_{xy}
\]

Plane stress (2-D): \( T = T (x, y), \quad \sigma_z = 0, \quad \tau_{yz} = 0, \quad \tau_{xz} = 0 \)

\[
\varepsilon_x = \frac{1}{E} \left( \sigma_x - \mu \sigma_y \right) + \alpha T , \\
\varepsilon_y = \frac{1}{E} \left( \sigma_y - \mu \sigma_x \right) + \alpha T , \\
\varepsilon_z = \frac{1}{E} \left[ - \mu (\sigma_x + \sigma_y) + \alpha \right] \\
\gamma_{xy} = \frac{2(1 + \mu)}{E} \tau_{xy}
\]
Plane Strain: \( T = T(x, y), \quad \varepsilon_z = 0, \quad \tau_{yz} = \tau_{xz} = 0, \)

\( \varepsilon_z = 0 \Rightarrow \sigma_z = \mu(\sigma_x + \sigma_y) - E\alpha T \)

\[
\begin{align*}
\varepsilon_x &= \frac{1 - \mu^2}{E} \left[ \sigma_x - \frac{\mu}{1 - \mu} \sigma_y \right] + (1 + \mu)\alpha T \\
\varepsilon_y &= \frac{1 - \mu^2}{E} \left[ \sigma_y - \frac{\mu}{1 - \mu} \sigma_x \right] + (1 + \mu)\alpha T \\
\gamma_{xy} &= \frac{2(1 + \mu)}{E} \tau_{xy}
\end{align*}
\]

Plane stress:

\[
\begin{array}{ccc}
E & \mu & \alpha \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{array}
\]

Plane strain:

\[
\begin{array}{ccc}
\frac{E}{1 - \mu^2} & \frac{\mu}{1 - \mu} & (1 + \mu)\alpha
\end{array}
\]

6.2 Solution in terms of displacements

We can express the physical equation of plane stress as:
\[
\begin{align*}
\sigma_x &= \frac{E}{1-\mu^2} (\varepsilon_x + \mu \varepsilon_y) - \frac{E \alpha T}{1-\mu}, \\
\sigma_y &= \frac{E}{1-\mu^2} (\varepsilon_y + \mu \varepsilon_x) - \frac{E \alpha T}{1-\mu}, \\
\tau_{xy} &= \frac{E}{2(1+\mu)} \gamma_{xy}.
\end{align*}
\]

After substitution of geometrical equations, we have
\[
\begin{align*}
\sigma_x &= \frac{E}{1-\mu^2} \left[ \frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial y} \right] - \frac{E \alpha T}{1-\mu}, \\
\sigma_y &= \frac{E}{1-\mu^2} \left[ \frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x} \right] - \frac{E \alpha T}{1-\mu}, \\
\tau_{xy} &= \frac{E}{2(1+\mu)} \left[ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right]
\end{align*}
\]

Substituting above into equilibrium equations and with \( X = Y = 0 \), we have:
\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{1-\mu}{2} \frac{\partial^2 u}{\partial y^2} + \frac{1+\mu}{2} \frac{\partial^2 v}{\partial x \partial y} - (1+\mu) \alpha \frac{\partial T}{\partial x} &= 0, \\
\frac{\partial^2 v}{\partial y^2} + \frac{1-\mu}{2} \frac{\partial^2 v}{\partial x^2} + \frac{1+\mu}{2} \frac{\partial^2 u}{\partial x \partial y} - (1+\mu) \alpha \frac{\partial T}{\partial y} &= 0.
\end{align*}
\]

The stress boundary conditions can also be expressed in terms of displacements, put \( X = Y = 0 \) for the case of no surface forces:
\[
\begin{align*}
&l \left[ \frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial y} \right]_s + m \frac{1-\mu}{2} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right]_s = l(1+\mu) \alpha T, \\
&m \left[ \frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x} \right]_s + l \frac{1-\mu}{2} \left[ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right]_s = m(1+\mu) \alpha T.
\end{align*}
\]
Comparing the above equations with those in chapter 2, we have the conclusion that:

The displacements components \( u \) and \( v \) produced by the temperature change \( T \) are the same as those produced by the body force components

\[
X = -\frac{E\alpha}{1-\mu} \frac{\partial T}{\partial x}, \quad Y = -\frac{E\alpha}{1-\mu} \frac{\partial T}{\partial y}
\]

together with the normal surface forces

\[
q_N = \frac{E\alpha T}{1-\mu}
\]

on the stress boundary. The displacement boundary condition is the same as before. Once the \( u, v \) are found, the thermal stress can be calculated from physical equations. Thus, a problem of thermal stresses is converted to an ordinary problem of stresses due to action of external loads.

For solution of plane strain thermal stress problems, we just take replacement of constants \( E, \mu, \alpha \) by \( \frac{E}{1-\mu^2}, \frac{\mu}{1-\mu}, (1+\mu)\alpha \).

### 6.3 Use of Displacement Potential

To solve the above differential equations and boundary conditions in terms of displacements, we separate the solution into two parts:

1. particular solution which satisfies equilibrium equations only;

2. complementary solution which, in combination with the particular solution, satisfies the boundary conditions.

<1> Particular Solution \( u', v' \)
We introduce a thermoelastic displacement potential function $\Psi(x, y)$, so that
\[
u' = \frac{\partial \Psi}{\partial x}, \quad \nu' = \frac{\partial \Psi}{\partial y}
\]
Substituting $u = u', v = v'$ into equilibrium equation, we obtain
\[
\frac{\partial}{\partial x} \left[ \nabla^2 \Psi \right] = (1 + \mu) \alpha \frac{\partial T}{\partial x}, \quad \frac{\partial}{\partial y} \left[ \nabla^2 \Psi \right] = (1 + \mu) \alpha \frac{\partial T}{\partial y}.
\]
Evidently, if $\Psi$ is a solution of
\[
\nabla^2 \Psi = (1 + \mu) \alpha T
\]
then $u' = \frac{\partial \Psi}{\partial x}, \quad v' = \frac{\partial \Psi}{\partial y}$ satisfies equilibrium equations and may be used as a particular solution.

The stress components corresponding to the particular solution of displacements are
\[
\sigma_x' = -\frac{E}{1 + \mu} \frac{\partial^2 \Psi}{\partial y^2}, \quad \sigma_y' = -\frac{E}{1 + \mu} \frac{\partial^2 \Psi}{\partial x^2}, \quad \tau_{xy}' = -\frac{E}{1 + \mu} \frac{\partial^2 \Psi}{\partial x \partial y}.
\]

<2> Complementary Solution $u''$, $v''$ must satisfy the homogeneous equations:

\[
\begin{align*}
\frac{\partial^2 u''}{\partial x^2} + \frac{1 - \mu}{2} \frac{\partial^2 u''}{\partial y^2} + \frac{1 + \mu}{2} \frac{\partial^2 v''}{\partial x \partial y} &= 0 \\
\frac{\partial^2 v''}{\partial y^2} + \frac{1 - \mu}{2} \frac{\partial^2 v''}{\partial x^2} + \frac{1 + \mu}{2} \frac{\partial^2 u''}{\partial x \partial y} &= 0
\end{align*}
\]
The corresponding stress components $\sigma_x''$, $\sigma_y''$, $\tau_{xy}''$ can be calculated (with $T=0$).

Thus the total displacements $u=u'+u''$, $v=v'+v''$ must satisfy the displacement boundary conditions; the total stress components $\sigma_x=\sigma_x'+\sigma_x''$, $\sigma_y=\sigma_y'+\sigma_y''$, $\tau_{xy}=\tau_{xy}'+\tau_{xy}''$ must satisfy the stress boundary condition.

In the case of stress boundary problems, the complementary stress components can be also found from stress function $\phi$. ($\phi$ must satisfies compatibility condition)

$$\begin{align*}
E &\rightarrow \frac{E}{(1-\mu^2)} \\
\mu &\rightarrow \frac{\mu}{(1-\mu)} \\
\alpha &\rightarrow (1+\mu)\alpha
\end{align*}$$

For plane strain problem, we just

$\diamondsuit$ An example:

Consider the rectangular plate subjected to a temperature change of $6^{\circ}$.
\[
T = T_0 \left(1 - \frac{y^2}{b^2}\right)
\]

\[T_0\] is constant

We have equation for \(\Psi\):

\[
\nabla^2 \Psi = (1 + \mu)\alpha T_0 \left(1 - \frac{y^2}{b^2}\right)
\]

which is easily satisfied by taking

\[
\Psi = (1 + \mu)\alpha T_0 \left[\frac{y^2}{2} - \frac{y^4}{12b^4}\right]
\]

The corresponding stress components are

\[
\sigma_x' = -E\alpha T_0 \left(1 - \frac{y^2}{b^2}\right), \quad \sigma_y' = 0, \quad \tau_{xy}' = 0.
\]

which does not satisfy the stress boundary condition at edges \(x = \pm a\). In order to exactly satisfy the boundary condition, surface force

\[
\bar{X} = -\sigma_x \bigg|_{x=\pm a} = E\alpha T_0 \left(1 - \frac{y^2}{b^2}\right)
\]

must be applied, and the corresponding stresses are the required \(\sigma_x''\), \(\sigma_y''\), \(\tau_{xy}''\). However, there are no simple analytic solutions for this.

When \(b << a\), then we can apply the Saint-Venant’s principle and replace the surface force \(\bar{X}\) by statically equivalent uniform tensile forces to get an approximate solution:

We take stress function \(\phi = cy^2\) which satisfy \(\nabla^4 \phi = 0\), thus

\[
\sigma_x'' = 2C, \quad \sigma_y'' = \frac{\partial^2 \phi}{\partial x^2} = 0, \quad \tau_{xy}'' = -\frac{\partial^2 \phi}{\partial x \partial y} = 0.
\]
The total stress components are given by superposition:

\[ \sigma_x = \sigma_x' + \sigma_x'' = 2C - E\alpha T_0 \left(1 - \frac{y^2}{b^2}\right) \]
\[ \sigma_y = 0, \quad \tau_{xy} = 0 \]

The boundary conditions are

\[ \sigma_x \bigg|_{x=\pm a} = 0, \quad \tau_{xy} \bigg|_{x=\pm a} = 0, \quad \sigma_y \bigg|_{y=\pm b} = 0, \quad \tau_{xy} \bigg|_{y=\pm b} = 0. \]

Satisfied

\[
\int_{-b}^{b} \sigma_x \bigg|_{x=\pm a} \, dy = 0, \quad \int_{-b}^{b} \sigma_x \bigg|_{x=\pm a} \, y \, dy = 0.
\]

\[
C = \frac{1}{3} E\alpha T_0
\]

The final solutions are

\[ \sigma_x = E\alpha T_0 \left[ \frac{y^2}{b^2} - \frac{1}{3} \right], \quad \sigma_y = 0, \quad \tau_{xy} = 0. \]

(near the two edges, the solution is approximate!)
\section*{6.4 Solution in Polar Coordinates}

\[ \varepsilon_r = \frac{1}{E} \left( \sigma_r - \mu \sigma_\theta \right) + \alpha T, \quad \varepsilon_\theta = \frac{1}{E} \left( \sigma_\theta - \mu \sigma_r \right) + \alpha T, \]

\textbf{Physical equations:}

\[ \gamma_{r\theta} = \frac{2(1 + \mu)}{E} \tau_{r\theta}. \]

\textbf{Thermoelastic displacement potential} \( \Psi = \Psi(r, \theta) \): The particular solution \( u_r', u_\theta' \) of equilibrium equation can be expressed as

\[ u_r' = \frac{\partial \Psi}{\partial r}, \quad u_\theta' = \frac{1}{r} \frac{\partial \Psi}{\partial \theta} \]

\( \Psi \) should satisfy

\[ \nabla^2 \Psi = (1 + \mu) \alpha T, \]

where \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \)

The corresponding stress components are

\[ \sigma_r' = -\frac{E}{1 + \mu} \left( \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} \right) \]

\[ \sigma_\theta' = -\frac{E}{1 + \mu} \frac{\partial^2 \Psi}{\partial r^2}, \quad \tau_{r\theta}' = \frac{E}{1 + \mu} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Psi}{\partial \theta} \right) \]

In case of \textbf{axisymmetrical temperature change} \( T(r) \), the displacement potential may be taken as \( \Psi(r) \), the above equation becomes

\[ \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right] \Psi = (1 + \mu) \alpha T \]

After integration, we have
\[ \Psi(r) = (1 + \mu) \alpha \int_{r}^{1} Tr(dr)^{2} + (1 + \mu) \alpha A \ln r + B \]

A and B are constants. The stress components become

\[ \sigma_{r}' = -\frac{E}{1 + \mu} \frac{1}{r} \frac{d\Psi}{dr} = -\frac{E\alpha}{r^2} \left[ \int Trdr + A \right], \]

\[ \sigma_{\theta}' = -\frac{E}{1 + \mu} \frac{d^2\Psi}{dr^2} = \frac{E\alpha}{r^2} \left[ \int Trdr + A - Tr^2 \right], \]

\[ \tau_{r\theta}' = 0. \]

It is better to change the indefinite integrals into definite integrals, we have

\[ \sigma_{r}' = -\frac{E\alpha}{r^2} \left[ \int_{\rho}^{r} Trdr + A \right], \]

\[ \sigma_{\theta}' = \frac{E\alpha}{r^2} \left[ \int_{\rho}^{r} Trdr + A - Tr^2 \right], \quad \tau_{r\theta}' = 0. \]

where \( \rho \) is an arbitrary quantity with a dimension of [length].

In case of plane strain, we have:

\[ E \rightarrow \frac{E}{1 - \mu}, \quad \mu \rightarrow \frac{\mu}{1 - \mu}, \quad \alpha \rightarrow (1 + \mu)\alpha, \]

\[ \sigma_{z} = \mu(\sigma_{r} + \sigma_{\theta}) - E\alpha T \]

♦ The complementary solution can be obtained by the same process as in rectangular coordinates.

### 6.5 Axisymmetrical Thermal Stresses in Circular Rings and Barrels

♦ Consider a circular ring with inner radius \( a \) and outer radius \( b \), subjected to axisymmetrical temperature change \( T = T(r) \). The boundary conditions are
\[
\begin{align*}
\sigma_r \big|_{r=a} &= 0, \quad \sigma_r \big|_{r=b} = 0 \\
\end{align*}
\]

Assuming a plane stress condition, and taking \( \rho = a \) in the particular solution (see above section), we have

\[
\begin{align*}
\sigma_r' &= -E \frac{a}{r^2} \left[ \int_a^r Trdr + A \right] \\
\sigma_\theta' &= E \frac{a}{r^2} \left[ \int_a^r Trdr + A - Tr^2 \right], \\
\tau_{r\theta}' &= 0.
\end{align*}
\]

Evidently, the above two boundary conditions \((\sigma_r \big|_{r=a} = 0, \quad \sigma_r \big|_{r=b} = 0)\) cannot be satisfied by any value of the single arbitrary constant \(A\). Hence we take a stress function \(\phi = \frac{C^2 r^2}{2}\) (satisfy \(\nabla^4 \phi = 0\) already), this gives the complementary stress components

\[
\begin{align*}
\sigma_r'' &= \sigma_\theta'' = C, \\
\tau_{r\theta}'' &= 0.
\end{align*}
\]

We can finally obtain the total stress components:

\[
\begin{align*}
\sigma_r &= E \frac{a}{r^2} \left[ \frac{r^2 - a^2}{b^2 - a^2} \int_a^b Trdr - \int_a^r Trdr \right], \\
\sigma_\theta &= E \frac{a}{r^2} \left[ \frac{r^2 + a^2}{b^2 - a^2} \int_a^b Trdr + \int_a^r Trdr - Tr^2 \right], \\
\tau_{r\theta} &= 0.
\end{align*}
\]

**Home work:***

(1) For a long circular barrel, assuming plane strain condition, determine the thermal stress components \(\sigma_r, \sigma_\theta, \tau_{r\theta}\) and \(\sigma_z\).

(2) For a barrel of finite length with free ends (i.e., \(\sigma_z = 0\) at the ends), determine the stress components according to Saint-Venant’s principle.

(3) For state of steady heat flow, the temperature change at any point is (by
theory of heat transfer)

\[ T = T_a \frac{\ln \left( \frac{b}{r} \right)}{\ln \left( \frac{b}{a} \right)} + T_b \frac{\ln \left( \frac{a}{r} \right)}{\ln \left( \frac{a}{b} \right)} \]

where \( T|_{r=a} = T_a \), \( T|_{r=b} = T_b \). determine the stress components \( \sigma_r \), \( \sigma_\theta \), \( \sigma_z \) (plane strain).